Algebraic Dynamical Systems and Dirichlet's Unit Theorem on Arithmetic Varieties

Huayi Chen[1] and Atsushi Moriwaki[2]

[1]Institut Fourier (UMR 5582), Université Grenoble Alpes, F-38402 Grenoble, France and [2]Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

Correspondence to be sent to: huayi.chen@ujf-grenoble.fr

The effectivity of pseudo-effective $\mathbb{R}$-divisor is an important and difficult problem in algebraic geometry. In the Arakelov geometry framework, this problem can be considered as a generalization of Dirichlet's unit theorem for number fields. In this article, we propose obstructions to the Dirichlet property by two approaches, that is, the denseness of nonpositive points and functionals on adelic $\mathbb{R}$-divisors. Applied to the algebraic dynamical systems, these results provide examples of nef adelic arithmetic $\mathbb{R}$-Cartier divisor which does not have the Dirichlet property. We hope the obstructions obtained in the article will give ways toward criteria of the Dirichlet property.

1 Introduction

Let X be a projective geometrically integral variety over a number field K and let $\bar{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X (for details of adelic arithmetic divisors, see [17]). We say that $\bar{D}$ has *the Dirichlet property* if $\bar{D} + \widehat{(\psi)}$ is effective for some $\psi \in \mathrm{Rat}(X)^\times_{\mathbb{R}} (:= \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R})$. It is clear that, if $\bar{D}$ has the Dirichlet property, then it is pseudo-effective, namely for any big adelic arithmetic $\mathbb{R}$-Cartier divisor

$\bar{E}$, the sum $\bar{D} + \bar{E}$ is also big. In [16], the following question has been proposed:

> Assume that $\bar{D}$ is pseudo-effective. Under which condition does it follow that $\bar{D}$ has the Dirichlet property?

In the case where $X = \operatorname{Spec} K$, the pseudo-effectivity actually implies the Dirichlet property. It can be considered as an Arakelov geometry interpretation of the classical Dirichlet's unit theorem. Therefore, the above problem could be seen as the study of possible higher dimensional generalizations of the Dirichlet's unit theorem. Note that the analog of the above problem in the algebraic geometry setting asks for effectivity criteria for pseudo-effective $\mathbb{R}$-divisors. Few results have been known except some special cases such as in [2, Lemma 6.1].

The Arakelov geometry approach is very suitable to study this effectivity problem. In fact, it is known that the above question has a positive answer in the following cases:

(1) $X = \operatorname{Spec}(K)$ (the classical Dirichlet's unit theorem).
(2) D is numerically trivial on X (cf. [16, 17]).
(3) X is a toric variety and $\bar{D}$ is of toric type (cf. [4]).

Moreover, in the arithmetic setting, the positivity of adelic divisors can be considered both horizontally and vertically. Thus, one has more flexibility in the study of the effectivity problem. For example, in the case where D is a pseudo-effective $\mathbb{R}$-Cartier divisor on X, the arithmetic pseudo-effectivity of $\bar{D}$ can be described by the positivity of a numerical function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ on $\bar{D}$ (see Proposition 8.2 *infra*). In the case where D is big, then $\bar{D}$ is arithmetically big if and only if this function valued on $\bar{D}$ is > 0. If $\bar{D}$ is pseudo-effective and D is big, then any positive perturbation of the Green functions of $\bar{D}$ leads to a big adelic arithmetic $\mathbb{R}$-Cartier divisor, which means that the multiples of $\bar{D}$ have many "almost small sections". Heuristically, one can obtain the effectivity of $\bar{D}$ by some limit procedure.

The aim of this paper is to discover the subtleties behind the above intuitive arguments and to study the obstructions to the Dirichlet property. We will construct from an algebraic dynamical system over a number field a nef adelic arithmetic Cartier divisor $\bar{D}$ which does not have the Dirichlet property. We observe from this construction that the obstruction comes from the denseness of the set of preperiodic points with respect to the analytic topology. More precisely, let $f : X \to X$ be a surjective endomorphism of X over K. Let D be an ample $\mathbb{R}$-Cartier divisor on X such that $f^*(D) = dD + (\varphi)$ for some real number $d > 1$ and $\varphi \in \operatorname{Rat}(X)_{\mathbb{R}}^{\times}$. Let $\bar{D} = (D, g)$ be the

canonical compactification of D, that is, $\bar{D}$ is an adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type with $f^*(\bar{D}) = d\bar{D} + \widehat{(\varphi)}$ (cf. Section 4). Note that $\bar{D}$ is nef (cf. Lemma 4.1). We establish the following theorem.

Theorem 1.1 (cf. Theorem 4.5). If the set of all preperiodic points of f is dense on some connected component of $X(\mathbb{C})$ with respect to the analytic topology (the topology as an analytic space), then the Dirichlet property of $\bar{D}$ does not hold.   □

The proof of the theorem relies on a necessary condition of the Dirichlet property established in Lemma 2.1. We actually prove that the essential support (see (2.1) for definition) of algebraic points with nonpositive heights should not meet the strictly effective locus of an effective section of the adelic arithmetic $\mathbb{R}$-Cartier divisor.

The concrete examples to apply the above theorem are discussed in Section 5. Even for the algebraic dynamical system as treated in Theorem 1.1, it is a very interesting and challenging problem to find a nontrivial sufficient condition to ensure the Dirichlet property. Further, in [18], we introduce a geometric analog of the above question. Namely, if D is a pseudo-effective $\mathbb{Q}$-Cartier divisor on a normal projective variety defined over a finite field, can we conclude that D is $\mathbb{Q}$-effective? It actually holds on a certain kind of abelian scheme over a curve, so that the situation of the geometric case is different from the arithmetic case.

Note that the essential support of a family S of algebraic points is not empty only if the family S is Zariski dense. Therefore, Lemma 2.1 provides nontrivial necessary conditions for the Dirichlet property only when the set of nonpositive points is Zariski dense. In order to obtain obstructions to the Dirichlet property for general adelic arithmetic $\mathbb{R}$-Cartier divisors, we propose the functional analysis approach. We introduce the notion of asymptotic maximal slope $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ for any adelic arithmetic $\mathbb{R}$-Cartier divisor on X (see Sections 6.2 and 8), which is the threshold of the Dirichlet property where we consider the twists of the adelic arithmetic $\mathbb{R}$-Cartier divisor by the pull-back of adelic arithmetic $\mathbb{R}$-Cartier divisors on $\mathrm{Spec} K$. We prove that this arithmetic invariant also determines the pseudo-effectivity of the adelic arithmetic $\mathbb{R}$-Cartier divisor (see Proposition 8.2). Therefore, the Dirichlet property and the pseudo-effectivity are naturally linked by this numerical invariant. We then obtain a necessary condition of the Dirichlet property in terms of the directional derivative of the asymptotic maximal slope, which is a functional on the space of all adelic arithmetic $\mathbb{R}$-Cartier divisors (nonnecessarily additive a priori).

Theorem 1.2. Let $\bar{D}$ be a pseudo-effective adelic arithmetic $\mathbb{R}$-Cartier divisor on X such that D is big. If $\bar{D}$ has the Dirichlet property, then either $\bar{D}$ is big, or $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = 0$ and for any place v of the number field K, the support of the super-additive functional Ψ_v on $C^0(X_v^{\mathrm{an}})_+$ defined by the directional derivatives of $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ at $\bar{D}$ cannot contain any algebraic subvariety of dimension $\geqslant 1$ except those embedded in the augmented base locus of D_v, where $C^0(X_v^{\mathrm{an}})_+$ denotes the space of nonnegative continuous functions on the analytification X_v^{an} (in the sense of Berkovich if v is a finite place). $\square$

For this purpose, we establish a general analysis for functionals on the spaces of adelic arithmetic $\mathbb{R}$-Cartier divisors as in Theorem 6.2. This result can be applied to not only the maximal asymptotic slope (see Corollary 6.7) but also other natural arithmetic invariants such as the arithmetic volume function (see Corollary 6.11) and the arithmetic self-intersection number (see Corollary 6.12). In Section 7, we compare these specifications of Theorem 6.2. The comparisons show that the arithmetic maximal slope is particularly adequate in the study of the Dirichlet property of pseudo-effective adelic arithmetic $\mathbb{R}$-Cartier divisors.

When D is a big $\mathbb{R}$-Cartier divisor, $\bar{D}$ is relatively nef and $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$ coincides with the normalized height of X with respect to $\bar{D}$ (typically it is the case when $\bar{D}$ arises from an arithmetic dynamical system as the canonical polarization), the functional Ψ_v identifies with the Monge–Ampère measure (in the sense of Chambert-Loir [5] when v is a finite place). Surprisingly, the condition on the support of the functional in the above theorem coincides with the key point in the Arakelov approach to the Bogomolov problem (see [11, 20, 23]). It is interesting to know if there is a deeper relation between Dirichlet and Bogomolov problems.

The article is organized as follows. After a description on the notation and terminology that we will use in the article, a necessary condition of the Dirichlet property is proved in the second section in the form of nondenseness for nonpositive algebraic points, which leads to an obstruction of the Dirichlet property in the setting of arithmetic dynamical systems (see Theorem 4.5) in Section 4, using the canonical Green function in algebraic dynamical systems discussed in Section 3. Concrete counter-examples are provided in Section 5. The functional approach to the study of the Dirichlet property is introduced in Section 6. The obstructions of the Dirichlet property are proposed in terms of the properties of some possibly nonlinear functionals. Section 7 is devoted to the comparison of these functionals, and also the comparison of the two approaches proposed in the article. We conclude the article in Section 8 by a refined version of the question above (see Question 8.4). We hope that our work will provide clues for the

further research on criteria of the Dirichlet property for higher dimension arithmetic variety.

1.1 Conventions and terminology

In this paper, we frequently use the same notation as in [15, 17].

1.1.1

Let V be a variety over a field F and $\bar{F}$ be an algebraic closure of F. Let x be an $\bar{F}$-valued point of V, that is, a morphism

$$x : \operatorname{Spec}(\bar{F}) \to V$$

over F. The residue field of V at the closed point given by the image of x is denoted by $F(x)$.

In the following, let X be a projective and geometrically integral scheme over a number field K. Let d be its Krull dimension.

1.1.2

Let O_K be the ring of integers in K and M_K be the set of all maximal ideals of O_K. Let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. For each $v \in M_K \cup K(\mathbb{C})$, we define K_v to be

$$K_v := \begin{cases} K \otimes_K^\sigma \mathbb{C} \text{ with respect to } \sigma & \text{if } v = \sigma \in K(\mathbb{C}), \\ \text{the completion of } K \text{ at } \mathfrak{p} & \text{if } v = \mathfrak{p} \in M_K. \end{cases}$$

Moreover, let X_v denote the fiber product $X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(K_v)$. Note that K_σ is naturally isomorphic to $\mathbb{C}$ via $a \otimes^\sigma z \mapsto \sigma(a)z$ and X_σ is nothing more than the fiber product $X \times_{\operatorname{Spec}(K)}^\sigma \operatorname{Spec}(\mathbb{C})$ with respect to σ.

Let x be a $\bar{K}$-valued point of X. Let $\{\phi_1, \ldots, \phi_n\}$ be the set of all K_v-algebra homomorphisms $K(x) \otimes_K K_v \to \bar{K}_v$, where $\bar{K}_v$ is an algebraic closure of K_v. Note that $n = [K(x) : K]$. For each $i = 1, \ldots, n$, let w_i be the $\bar{K}_v$-valued point of X_v given by the composition of morphisms

$$\operatorname{Spec}(\bar{K}_v) \xrightarrow{\phi_i^a} \operatorname{Spec}(K(x) \otimes_K K_v) \xrightarrow{x \times \operatorname{id}_{K_v}} X_v,$$

where ϕ_i^a is the morphism of K_v-schemes induced by ϕ_i. We denote $\{w_1, \ldots, w_n\}$ by $O_v(x)$.

1.1.3

For $v \in M_K \cup K(\mathbb{C})$, the analytification X_v^{an} of X_v is defined by

$$X_v^{\mathrm{an}} := \begin{cases} X_\sigma(K_\sigma) & \text{if } v = \sigma \in K(\mathbb{C}), \\ X_{\mathfrak{p}}^{\mathrm{an}} \text{ in the sense of Berkovich [1]} & \text{if } v = \mathfrak{p} \in M_K. \end{cases}$$

As K_σ is naturally identified with $\mathbb{C}$, $X_\sigma^{\mathrm{an}} = X_\sigma(K_\sigma) = X_\sigma(\mathbb{C})$. We equip the space X_σ^{an} (respectively, $X_{\mathfrak{p}}^{\mathrm{an}}$) with the analytic topology, namely the topology as an analytic space (respectively, as a Berkovich space). Let X_∞^{an} denote the set of $\mathbb{C}$-valued points of X over $\mathbb{Q}$. Note that $X_\infty^{\mathrm{an}} = \coprod_{\sigma \in K(\mathbb{C})} X_\sigma^{\mathrm{an}}$. We often denote X_∞^{an} by $X(\mathbb{C})$. Note that the complex conjugation induces an involution $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.

1.1.4

Let us fix $v \in M_K \cup K(\mathbb{C})$. For a $\bar{K}_v$-valued point w of X_v, we define $w^{\mathrm{an}} \in X_v^{\mathrm{an}}$ to be

$$w^{\mathrm{an}} := \begin{cases} w & \text{if } v = \sigma \in K(\mathbb{C}), \\ \text{the valuation of } K_{\mathfrak{p}}(w) \text{ as an extension of } v_{\mathfrak{p}} & \text{if } v = \mathfrak{p} \in M_K, \end{cases}$$

where $v_{\mathfrak{p}}$ is the valuation of $K_{\mathfrak{p}}$ defined as $v_{\mathfrak{p}}(f) = \#(O_K/\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(f)}$. Note that

$$\#\{w' \in X_v(\bar{K}_v) \mid w'^{\mathrm{an}} = w^{\mathrm{an}}\} = [K_v(w) : K_v].$$

1.1.5

Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X and denote by $\mathrm{Div}_{\mathbb{R}}(X)$ the $\mathbb{R}$-vector space $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. If φ is an element in $\mathrm{Rat}(X)^\times$, we denote by (φ) its divisor, which is an element in $\mathrm{Div}(X)$. The Cartier divisors on X constructed in this way are called *principal divisors*. The map $\varphi \mapsto (\varphi)$ is a group homomorphism and extends by extension of scalar to an $\mathbb{R}$-linear map $\mathrm{Rat}(X)_{\mathbb{R}}^\times \to \mathrm{Div}_{\mathbb{R}}(X)$, where $\mathrm{Rat}(X)_{\mathbb{R}}^\times := \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$.

By an *adelic arithmetic $\mathbb{R}$-Cartier divisor* of C^0-type on X, we refer to a pair $\bar{D} = (D, g)$, where $D \in \mathrm{Div}_{\mathbb{R}}(X)$ and $g = (g_v)_{v \in M_K \cup K(\mathbb{C})}$ is a family of Green functions, with g_v being a D-Green function of C^0-type on X_v^{an}. We also require that $g_{\mathfrak{p}}$ comes from an integral model of D for all but a finite number of $\mathfrak{p} \in M_K$, and that the family $(g_\sigma)_{\sigma \in K(\mathbb{C})}$ is invariant under the action of F_∞. The family $g = (g_v)_{v \in M_K \cup K(\mathbb{C})}$ is often denoted by

$$\sum_{v \in M_K \cup K(\mathbb{C})} g_v[v].$$

If it is not specified, an adelic arithmetic $\mathbb{R}$-Cartier divisor refers to an adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. We denote the vector space consisting of adelic arithmetic $\mathbb{R}$-Cartier divisors on X by $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$. If φ is an element in $\mathrm{Rat}(X)^{\times}$, we define an adelic arithmetic $\mathbb{R}$-Cartier divisor as follows:

$$\widehat{(\varphi)} := \left((\varphi), \sum_{v \in M_K \cup K(\mathbb{C})} -\log |\varphi_v|_v^2[v] \right),$$

where φ_v is the rational function on X_v^{an} induced by φ. The map $\mathrm{Rat}(X)^{\times} \to \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ extends naturally to $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ and defines an $\mathbb{R}$-linear homomorphism of vector spaces. Any element in the image of this $\mathbb{R}$-linear map is called a *principal* adelic arithmetic $\mathbb{R}$-Cartier divisor.

For any $v \in M_K \cup K(\mathbb{C})$, one has a natural embedding from the vector space $C^0(X_v^{\mathrm{an}})$ into $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ which sends $f_v \in C^0(X_v^{\mathrm{an}})$ to

$$\begin{cases} (0, \, f_v[v]) & \text{if } v \in M_K, \\ (0, \, \tfrac{1}{2} f_v[v] + \tfrac{1}{2} F_{\infty}^*(f_v)[\bar{v}]) & \text{if } v \in K(\mathbb{C}). \end{cases}$$

We denote by $\bar{O}(f_v)$ this adelic arithmetic $\mathbb{R}$-Cartier divisor.

In the particular case where $X = \mathrm{Spec}\,K$, an adelic arithmetic $\mathbb{R}$-Cartier divisor ζ on $\mathrm{Spec}\,K$ is a vector $(\zeta_v)_{v \in M_K \cup K(\mathbb{C})}$ in $\mathbb{R}^{M_K \cup K(\mathbb{C})} := \mathrm{Map}(M_K \cup K(\mathbb{C}), \mathbb{R})$ which we can write into the form of a formal sum

$$\sum_{v \in M_K \cup K(\mathbb{C})} \zeta_v[v],$$

where $\zeta_v = 0$ for all but a finite number of indices v. The *Arakelov degree* of ζ is then defined as

$$\widehat{\deg}(\zeta) = \frac{1}{2} \sum_{v \in M_K \cup K(\mathbb{C})} \zeta_v.$$

For more details of $\widehat{\deg}(.)$, see [17, Section 4.2]

1.1.6

Let $\bar{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. For any algebraic point x of X outside the support of D, the *normalized height* $h_{\bar{D}}(x)$ of x with respect

to $\bar{D}$ is defined to be

$$h_{\bar{D}}(x) := \frac{\widehat{\deg}(\bar{D}|_x)}{[K(x):K]} = \frac{1}{[K(x):K]} \sum_{v \in M_K \cup K(\mathbb{C})} \sum_{w \in O_v(x)} \frac{1}{2} g_v(w^{\mathrm{an}}).$$

This function can be extended to the set of all points in $X(\bar{K})$, see [17, Section 4.2] for details. A $\bar{K}$-valued point x of X is said to be *nonpositive with respect to* $\bar{D}$ if $h_{\bar{D}}(x) \leq 0$. Note that the height function $h_{\bar{D}}(.)$ does not change if we replace $\bar{D}$ by $\bar{D} + \widehat{(\phi)}$ with $\phi \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$. This is a consequence of the product formula for the number field K.

For any real number λ, we denote by $X(\bar{K})_{\leq\lambda}^{\bar{D}}$ the set of all $\bar{K}$-valued points of X whose height with respect to $\bar{D}$ is bounded from above by λ, namely

$$X(\bar{K})_{\leq\lambda}^{\bar{D}} := \{x \in X(\bar{K}) \mid h_{\bar{D}}(x) \leq \lambda\}.$$

The *essential minimum* of the height function $h_{\bar{D}}(\cdot)$ is defined as

$$\hat{\mu}_{\mathrm{ess}}(\bar{D}) := \inf\{\lambda \in \mathbb{R} \mid X(\bar{K})_{\leq\lambda}^{\bar{D}} \text{ is Zariski dense}\}.$$

The function $\hat{\mu}_{\mathrm{ess}}(.)$ takes value in $\mathbb{R} \cup \{-\infty\}$. Note that if $\bar{D}$ verifies the Dirichlet property, then the essential minimum of $h_{\bar{D}}(.)$ is nonnegative.

1.1.7

We say that an adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D} = (D, g)$ is *relatively nef* if D is a nef $\mathbb{R}$-Cartier divisor and g_v is of $(C^0 \cap \mathrm{PSH})$-type (for details, see [17, Section 2.1]). To each family $(\bar{D}_i)_{i=1}^{d+1}$ of relatively nef adelic arithmetic $\mathbb{R}$-Cartier divisor, one can associate a real number $\widehat{\deg}(\bar{D}_1 \cdots \bar{D}_{d+1})$ as in [17, Section 4.5] (the smoothness condition for the scheme X in *loc. cit.* is actually not necessary). The intersection number function $(\bar{D}_1, \ldots, \bar{D}_{d+1}) \mapsto \widehat{\deg}(\bar{D}_1, \ldots, \bar{D}_{d+1})$ is symmetric, additive, and $\mathbb{R}_+$-homogeneous in each coordinate and hence extends to a $(d+1)$-linear form on the vector space of integrable adelic arithmetic $\mathbb{R}$-Cartier divisors (recall that an adelic arithmetic $\mathbb{R}$-Cartier divisor is said to be *integrable* if it can be written as a difference of two relatively nef adelic arithmetic $\mathbb{R}$-Cartier divisors). The extended function is continuous in each of its coordinates with respect to the topology on the vector space of all integrable adelic arithmetic $\mathbb{R}$-Cartier divisors defined by the usual convergence in each of its finite-dimensional vector subspaces and the uniform convergence of Green functions. Therefore, for fixed integrable adelic arithmetic $\mathbb{R}$-Cartier divisors $\bar{D}_1, \ldots, \bar{D}_d$, the function $\bar{D}_{d+1} \mapsto \widehat{\deg}(\bar{D}_1, \ldots, \bar{D}_{d+1})$ can be extended by continuity to the whole vector space of adelic arithmetic $\mathbb{R}$-Cartier divisors on X.

Let $\bar{D}_1, \ldots, \bar{D}_d$ be relatively nef adelic arithmetic $\mathbb{R}$-Cartier divisors. The intersection product defines a (nonnegative) Radon measure $(\bar{D}_1 \cdots \bar{D}_d)_v$ on X_v^{an} for each place $v \in M_K \cup K(\mathbb{C})$ such that, for any $\phi \in C^0(X_v^{\mathrm{an}})$ one has

$$(\bar{D}_1 \cdots \bar{D}_d)_v(\phi) := \widehat{\deg}(\bar{D}_1 \cdots \bar{D}_d \cdot \bar{O}(\phi)).$$

More generally, one can define a signed Borel measure $(\bar{D}_1 \cdots \bar{D}_d)_v$ on X_v^{an} for integrable adelic arithmetic $\mathbb{R}$-Cartier divisors $\bar{D}_1, \ldots, \bar{D}_d$. Moreover, this signed measure is multi-linear in $\bar{D}_1, \ldots, \bar{D}_d$. Note that in the case where $\bar{D}_1, \ldots, \bar{D}_d$ come from adelic line bundles and v is a nonarchimedean place, the above measure has been constructed in [5] (the archimedean case is more classical and relies on the theory of Monge–Ampère operators). See Section 7.2 *infra.* for the integrability of Green functions with respect to this measure extending some results of [6, 14].

Let r be an integer in $\{0, \ldots, d\}$ and $\bar{D}_1, \ldots, \bar{D}_{r+1}$ be a family of integrable adelic arithmetic $\mathbb{R}$-Cartier divisors. If Z is an $\mathbb{R}$-coefficient algebraic cycle of dimension r in X, written into the linear combination of prime cycles as

$$Z = \lambda_1 Z_1 + \cdots + \lambda_n Z_n,$$

then we define the height of Z with respect to $\bar{D}_1, \ldots, \bar{D}_{r+1}$ as

$$h(\bar{D}_1, \ldots, \bar{D}_{r+1}; Z) := \sum_{i=1}^{n} \lambda_i \widehat{\deg}(\bar{D}_1|_{Z_i} \cdots \bar{D}_{r+1}|_{Z_i}).$$

In the particular case where all $\bar{D}_i$ are equal to the same adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D}$, we write $h(\bar{D}_0, \ldots, \bar{D}_r; Z)$ in abbreviation as $h(\bar{D}; Z)$. Note that when Z is the algebraic cycle corresponding to a closed point x of X, the height $h(\bar{D}; Z)$ equals $[K(x) : K]h_{\bar{D}}(x)$.

We say that an adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D} = (D, g)$ is *nef* if it is relatively nef and if the function $h_{\bar{D}}(\cdot)$ is nonnegative. In the case where $\bar{D}$ is nef, the function $h(\bar{D}; \cdot)$ is nonnegative on effective cycles.

1.1.8

Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X. If $s \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ and $D + (s) \geq 0$, then $|s|_{g_v} := |s|_v \exp(-g_v/2)$ is a continuous function on X_v^{an}, where $v \in M_K \cup K(\mathbb{C})$; $|s|_{g_v} \leq 1$ for all v if and only if $\bar{D} + \widehat{(s)} \geq 0$.

We denote by $H^0(X, D)$ the K-vector space

$$\{\phi \in \mathrm{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}.$$

Assume that s is an element in $H^0(X, D)$. For each $v \in M_K \cup K(\mathbb{C})$, the Green function g_v defines a continuous function $|s|_{g_v}$ such that

$$|s|_{g_v} = |s|_v \exp(-g_v/2).$$

This function vanishes on the locus of $\mathrm{div}(s) + D$. We also define

$$\|s\|_{v,\sup} := \sup_{x \in X_v^{\mathrm{an}}} |s|_v(x).$$

Denote by $\hat{H}^0(X, \bar{D})$ the set of all $s \in H^0(X, D)$ such that $\|s\|_{v,\sup} \leq 1$ for any $v \in M_K \cup K(\mathbb{C})$. The *arithmetic volume* of $\bar{D}$ is defined as

$$\widehat{\mathrm{vol}}(\bar{D}) := \limsup_{n \to +\infty} \frac{\log \#\hat{H}^0(X, n\bar{D})}{n^{d+1}/(d+1)!}.$$

1.1.9

Let V be an integral projective variety over a field F, and D be an $\mathbb{R}$-Cartier divisor on V. The *augmented base locus* of D is defined as the intersection of Zariski closed sets

$$\bigcap_{D=A+E} \mathrm{Supp}(E),$$

where the intersection is taken over all decompositions $D = A + E$ where A and E are, respectively, ample and effective $\mathbb{R}$-divisors on V. Note that if W is an integral closed subscheme of V of dimension > 0 which is not contained in the augmented base locus of D, then the restricted volume of D on X is strictly positive. We refer the readers to [10] for more details.

2   Denseness of Nonpositive Points

This section is devoted to a nondenseness result for nonpositive points under the Dirichlet property. This result will be useful in the following sections to construct counter-examples to the Dirichlet property. We fix a projective and geometrically integral scheme X defined over a number field K.

Let S be a subset of $X(\bar{K})$. For any closed subscheme Y of X such that $Y \subsetneq X$ and $v \in M_K \cup K(\mathbb{C})$, we set (see Sections 1.1.2 and 1.1.4)

$$\Delta(S; Y)_v^{\mathrm{an}} := \bigcup_{x \in S \setminus Y(\bar{K})} \{w^{\mathrm{an}} \mid w \in O_v(x)\}.$$

The essential support $\mathrm{Supp}_{\mathrm{ess}}(S)_v^{\mathrm{an}}$ of S at v is defined to be

$$\mathrm{Supp}_{\mathrm{ess}}(S)_v^{\mathrm{an}} := \bigcap_{Y \subsetneq X} \overline{\Delta(S; Y)_v^{\mathrm{an}}}, \tag{2.1}$$

where $\overline{\Delta(S; Y)_v^{\mathrm{an}}}$ is the closure of $\Delta(S; Y)_v^{\mathrm{an}}$ with respect to the analytic topology. Note that if $\Delta(S; \emptyset)_v^{\mathrm{an}}$ is dense with respect to the analytic topology, then

$$\mathrm{Supp}_{\mathrm{ess}}(S)_v^{\mathrm{an}} = X_v^{\mathrm{an}}.$$

Moreover, if S is not Zariski dense, then $\mathrm{Supp}_{\mathrm{ess}}(S)_v^{\mathrm{an}} = \emptyset$.

Lemma 2.1 (Nondenseness of nonpositive points). Let $\bar{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. If s is an element of $\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ with $\bar{D} + \widehat{(s)} \geq 0$, then

$$\mathrm{Supp}_{\mathrm{ess}}(X(\bar{K})_{\leq 0}^{\bar{D}})_v^{\mathrm{an}} \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} = \emptyset$$

for all $v \in M_K \cup K(\mathbb{C})$. In particular, if $\mathrm{Supp}(D + (s)) \neq \emptyset$, then $\Delta(X(\bar{K})_{\leq 0}^{\bar{D}}; \emptyset)_v^{\mathrm{an}}$ is not dense with respect to the analytic topology. $\qquad\square$

Proof. We set $S := X(\bar{K})_{\leq 0}^{\bar{D}}$, $Y := \mathrm{Supp}(D + (s))$ and $g'_v := -\log |s|_{g_v}^2$. By our assumption, $g'_v \geq 0$ for all $v \in M_K \cup K(\mathbb{C})$.

Claim 2.2. For all $y \in \Delta(S; Y)_v^{\mathrm{an}}$, we have $g'_v(y) = 0$. $\qquad\square$

Proof. For $y \in \Delta(S; Y)_v^{\mathrm{an}}$, we choose $x \in S \setminus Y(\bar{K})$ such that $y = w^{an}$ for some $w \in O_v(x)$. Then,

$$0 \geq 2[K(x) : K]h_{\bar{D}}(x) = 2\widehat{\deg}(\bar{D} + \widehat{(s)}|_x) = \sum_{v' \in M_K \cup K(\mathbb{C})} \sum_{w' \in O_{v'}(x)} g'_{v'}(w'^{\mathrm{an}}).$$

As $g'_{v'} \geq 0$ for all $v' \in M_K \cup K(\mathbb{C})$, the assertion follows. $\qquad\blacksquare$

We assume that $\mathrm{Supp}_{\mathrm{ess}}(S)_v^{\mathrm{an}} \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} \neq \emptyset$. In particular,

$$\overline{\Delta(S; Y)_v^{\mathrm{an}}} \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} \neq \emptyset.$$

We can choose $y_\infty \in X_v^{\mathrm{an}}$ and a sequence $\{y_m\}$ in X_v^{an} such that $|s|_{g_v}(y_\infty) < 1$, $y_m \in \Delta(S; Y)_v^{\mathrm{an}}$ for all m, and $\lim_{m \to \infty} y_m = y_\infty$. By the above claim, $|s|_{g_v}(y_m) = 1$ for all m, and hence $|s|_{g_v}(y_\infty) = \lim_{m \to \infty} |s|_{g_v}(y_m) = 1$. This is a contradiction. ∎

Remark 2.3. Let $f : \mathbb{P}_K^1 \to \mathbb{P}_K^1$ be a surjective endomorphism over K with $\deg(f) \geq 2$. Let S be the set of periodic points in $\mathbb{P}^1(\bar{K})$. Fix $\sigma \in K(\mathbb{C})$. By proposition 4.3 (which is also valid for the periodic points), $\Delta(S; \emptyset)_\sigma^{\mathrm{an}}$ coincides with the set of periodic points of f_σ in $(\mathbb{P}_K^1)_\sigma^{\mathrm{an}}$. The following are well known:

(1)   The closure of the set of repelling periodic points is the Julia set J_σ of f_σ [19, Theorem 4.2.10].

(2)   The set of nonrepelling periodic points is a finite set [19, Theorem 4.2.9].

(3)   The Julia set J_σ is closed and perfect, that is, J_σ is closed and J_σ has no isolated points in J_σ [19, Theorem 2.3.6].

Therefore, we can see that the essential support $\mathrm{Supp}_{\mathrm{ess}}(S)_\sigma^{\mathrm{an}}$ of S at σ is equal to the Julia set J_σ. □

For a subset S of $X(\bar{K})$ and $v \in M_K \cup K(\mathbb{C})$, we set $S_v = \bigcup_{x \in S} O_v(x)$. Let us consider a way to give the essential support of S at v in terms of S_v and X_v.

Proposition 2.4. $\mathrm{Supp}_{\mathrm{ess}}(S)_v^{\mathrm{an}} = \bigcap_{Z \subsetneq X_v} \overline{\{w^{\mathrm{an}} \mid w \in S_v \setminus Z(\bar{K}_v)\}}$, where Z runs over the set of all closed subschemes of X_v such that $Z \subsetneq X_v$. □

Proof.   It is sufficient to show that, for a closed subscheme Z of X_v such that $Z \subsetneq X_v$, there is a closed subscheme Y of X such that $Y \subsetneq X$ and that

$$\bigcup_{x \in S \setminus Y(\bar{K})} O_v(x) \subseteq S_v \setminus Z(\bar{K}_v),$$

that is, $S_v \cap Z(\bar{K}_v) \subseteq \bigcup_{x \in S \cap Y(\bar{K})} O_v(x)$.

Let $\pi : X_v \to X$ be the projection. For $x \in X(\bar{K})$ and $w \in X_v(\bar{K}_v)$, the natural induced morphisms

$$\mathrm{Spec}(\bar{K}) \to X_{\bar{K}} \quad \text{and} \quad \mathrm{Spec}(\bar{K}_v) \to (X_v)_{\bar{K}_v}$$

are denoted by $\tilde{x}$ and $\tilde{w}$, respectively, where

$$X_{\bar{K}} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K}) \quad \text{and} \quad (X_v)_{\bar{K}_v} = X_v \times_{\mathrm{Spec}(K_v)} \mathrm{Spec}(\bar{K}_v).$$

We fix an embedding $\bar{K} \hookrightarrow \bar{K}_v$, which yields a morphism $\bar{\pi} : (X_v)_{\bar{K}_v} \to X_{\bar{K}}$. In the case where $w \in O_v(x)$, there is a homomorphism $\iota_w : \bar{K} \to \bar{K}_v$ over K such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Spec}(\bar{K}_v) & \xrightarrow{\tilde{w}} & (X_v)_{\bar{K}_v} \\
\iota_w^a \downarrow & & \downarrow \bar{\pi} \\
\mathrm{Spec}(\bar{K}) & \xrightarrow{\tilde{x}} & X_{\bar{K}}
\end{array}$$

Let $\tilde{K}$ be the algebraic closure of K in $\bar{K}_v$. Note that $\iota_w(\bar{K}) = \tilde{K}$.

Let D be a Cartier divisor on X such that $X^\circ := X \setminus \mathrm{Supp}(D)$ is affine. Let A be a finitely generated K-algebra with $X^\circ = \mathrm{Spec}(A)$. Note that

$$S_v \cap \mathrm{Supp}(D)_v(\bar{K}_v) = \bigcup_{x \in S \cap \mathrm{Supp}(D)(\bar{K})} O_v(x).$$

If $Z \subseteq \mathrm{Supp}(D)_v$, then the assertion is obvious, so that we may assume that $Z \nsubseteq \mathrm{Supp}(D)_v$. We put $T = S_v \cap (Z(\bar{K}_v) \setminus \mathrm{Supp}(D)_v(\bar{K}_v))$.

Claim 2.5. There is a nonzero $h \in A \otimes_K \bar{K}$ such that $\tilde{w}^*(\bar{\pi}^*(h)) = 0$ for all $w \in T$. $\square$

Proof.   Let I_v be the ideal of $A \otimes_K K_v$ defining $Z \cap X_v^\circ$. Choose a nonzero element h' of I_v. There are $h_1, \ldots, h_r \in A \otimes_K \bar{K}$ and $a_1, \ldots, a_r \in \bar{K}_v$ such that

$$h' = a_1 \bar{\pi}^*(h_1) + \cdots + a_r \bar{\pi}^*(h_r)$$

and $a_1, \ldots, a_r$ are linearly independent over $\tilde{K}$. For $w \in T$ and $w \in O_v(x)$, by using the above diagram,

$$0 = \tilde{w}^*(h') = a_1 \tilde{w}^*(\bar{\pi}^*(h_1)) + \cdots + a_r \tilde{w}^*(\bar{\pi}^*(h_r)) = a_1 \iota_w(\tilde{x}^*(h_1)) + \cdots + a_r \iota_w(\tilde{x}^*(h_r)),$$

so that $\tilde{x}^*(h_1) = \cdots = \tilde{x}^*(h_r) = 0$. Therefore, the assertion follows.   ∎

We set $h = c_1 h_1 + \cdots + c_l h_l$ for some $c_1, \ldots, c_l \in \bar{K}$ and $h_1, \ldots, h_l \in A$. Let K' be a finite Galois extension of K such that $K(c_1, \ldots, c_l) \subseteq K'$. Here we put $f = \prod_{\sigma \in \mathrm{Gal}(K'/K)} \sigma(h)$. Note that $f \in A \setminus \{0\}$ and $w^*(\pi^*(f)) = 0$ for all $w \in T$, so that $T \subseteq \bigcup_{S \cap \mathrm{Spec}(A/fA)(\bar{K})} O_v(x)$. Therefore, if we set $Y = \mathrm{Supp}(D) \cup \mathrm{Spec}(A/fA)$, then the proposition follows.   ∎

3   Endomorphism and Green Function

This section consists of the construction of the canonical Green functions for a given $\mathbb{R}$-Cartier divisor in the algebraic dynamical system setting, which can be considered as a generalization of the construction of the canonical metrics in [22]. Here we explain them in terms of Green functions on either Berkovich spaces or complex varieties. Throughout this section, we fix the following notation. Let X be a projective and geometrically integral variety over a field K. Let $f : X \to X$ be a surjective endomorphism of X over K. Let D be an $\mathbb{R}$-Cartier divisor on X. We assume that there are a real number d and $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that $d > 1$ and $f^*(D) = dD + (\varphi)$.

3.1   Nonarchimedean case

We assume that K is the quotient field of a complete discrete valuation ring R. Let X^{an} be the analytification of X in the sense of Berkovich. Note that $f^{\mathrm{an}} : X^{\mathrm{an}} \to X^{\mathrm{an}}$ is also surjective by [1, Proposition 3.4.7]. Let us begin with the following proposition.

Proposition 3.1.   There exists a unique D-Green function g of C^0-type on X^{an} such that $(f^{\mathrm{an}})^*(g) = dg - \log |\varphi|^2$. □

Proof.   Let us fix a D-Green function g_0 of C^0-type on X^{an}. As

$$f^*(D) = dD + (\varphi),$$

$(f^{\mathrm{an}})^*(g_0)$ is a $(dD + (\varphi))$-Green function of C^0-type, and hence, there is a continuous function λ_0 on X^{an} such that

$$(f^{\mathrm{an}})^*(g_0) = dg_0 - \log |\varphi|^2 + \lambda_0.$$

For each $n \in \mathbb{Z}_{\geq 1}$, let us consider a continuous function h_n on X^{an} given by

$$h_n := \sum_{i=0}^{n} \frac{1}{d^{i+1}} ((f^{\mathrm{an}})^i)^*(\lambda_0).$$

Claim 3.2.

   (a)   There is a continuous function h on X^{an} such that the sequence $\{h_n\}$ converges uniformly to h.

   (b) $(f^{\mathrm{an}})^*(h) + \lambda_0 = dh$.   ∎

Proof. (a) It is sufficient to show that $\|h_n - h_m\|_{\sup} \to 0$ as $n, m \to \infty$. Indeed, if $n > m$, then

$$\|h_n - h_m\|_{\sup} = \left\| \sum_{i=m+1}^{n} \frac{1}{d^{i+1}} ((f^{\mathrm{an}})^i)^*(\lambda_0) \right\|_{\sup} \leq \sum_{i=m+1}^{n} \frac{1}{d^{i+1}} \left\| ((f^{\mathrm{an}})^i)^*(\lambda_0) \right\|_{\sup}$$

$$= \|\lambda_0\|_{\sup} \sum_{i=m+1}^{n} \frac{1}{d^{i+1}},$$

as required.

(b) Note that

$$(f^{\mathrm{an}})^*(h_n) + \lambda_0 = \sum_{i=0}^{n} \frac{1}{d^{i+1}} ((f^{\mathrm{an}})^{i+1})^*(\lambda_0) + \lambda_0 = dh_{n+1},$$

and hence the assertion follows. ∎

If we set $g = g_0 + h$, then g is a D-Green function of C^0-type and

$$(f^{\mathrm{an}})^*(g) = (f^{\mathrm{an}})^*(g_0) + (f^{\mathrm{an}})^*(h) = (dg_0 - \log|\varphi|^2 + \lambda_0) + (dh - \lambda_0) = dg - \log|\varphi|^2,$$

as desired.

Next we consider the uniqueness of g. Let g' be another D-Green function of C^0-type such that $(f^{\mathrm{an}})^*(g') = dg' - \log|\varphi|^2$. Then, as $g' - g$ is a continuous function on X^{an} and $(f^{\mathrm{an}})^*(g' - g) = d(g' - g)$, we have

$$\|g' - g\|_{\sup} = \|(f^{\mathrm{an}})^*(g' - g)\|_{\sup} = \|d(g' - g)\|_{\sup} = d\|g' - g\|_{\sup},$$

and hence $\|g' - g\|_{\sup} = 0$. Thus, $g' = g$.

Proposition 3.3. Let $\mathscr{X} \to \mathrm{Spec}(R)$ be a model of X over $\mathrm{Spec}(R)$, namely a projective and flat R-scheme whose generic fiber identifies with X, and $\mathscr{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$ such that $\mathscr{D}$ coincides with D on X. If there is an endomorphism $\tilde{f} : \mathscr{X} \to \mathscr{X}$ over $\mathrm{Spec}(R)$ such that $\tilde{f}\big|_X = f$ and $\tilde{f}^*(\mathscr{D}) = d\mathscr{D} + (\varphi)_{\mathscr{X}}$, then the D-Green function $g_{(\mathscr{X},\mathscr{D})}$ arising from the model $(\mathscr{X}, \mathscr{D})$ is equal to g. □

Proof. The relation $\tilde{f}^*(\mathscr{D}) = d\mathscr{D} + (\varphi)_{\mathscr{X}}$ yields

$$(f^{\mathrm{an}})^*(g_{(\mathscr{X},\mathscr{D})}) = dg_{(\mathscr{X},\mathscr{D})} - \log|\varphi|^2,$$

so that, by the uniqueness of g, we have $g_{(\mathscr{X},\mathscr{D})} = g$. ∎

Using the identities

$$f^*(D) = dD + (\varphi) \quad \text{and} \quad (f^{\mathrm{an}})^*(g) = g - \log|\varphi|^2,$$

we can easily see that

$$(f^n)^*(D) = d^n D + (\varphi_n) \quad \text{and} \quad ((f^{\mathrm{an}})^n)^*(g) = d^n g - \log|\varphi_n|^2$$

for $n \geq 1$, where

$$\varphi_n := \prod_{i=0}^{n-1} ((f^{n-1-i})^*(\varphi))^{d^i}.$$

Let $\mathscr{X} \to \mathrm{Spec}(R)$ be a model of X over $\mathrm{Spec}(R)$ and $\mathscr{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$ with $\mathscr{D}|_X = D$. For each $n \geq 1$, we choose a model $\mathscr{X}_n \to \mathrm{Spec}(R)$ of X over $\mathrm{Spec}(R)$ together with a morphism $\tilde{f}_n \colon \mathscr{X}_n \to \mathscr{X}$ over $\mathrm{Spec}(R)$ such that $\tilde{f}_n\big|_X = f^n$. Here we define an $\mathbb{R}$-Cartier divisor $\mathscr{D}_n$ on $\mathscr{X}_n$ to be

$$\mathscr{D}_n := \frac{1}{d^n}(\tilde{f}_n^*(\mathscr{D}) - (\varphi_n)_{\mathscr{X}_n}).$$

Note that $\mathscr{D}_n|_X = D$. Then we have the following proposition.

Proposition 3.4. If we set $\theta_n = g - g_{(\mathscr{X}_n, \mathscr{D}_n)}$, then $\lim_{n \to \infty} \|\theta_n\|_{\sup} = 0$. In particular, if $\mathscr{D}$ is relatively nef, then g is of $(C^0 \cap \mathrm{PSH})$-type. $\qquad\qquad\square$

Proof.   Since

$$\tilde{f}_n^*(\mathscr{D}) = d^n \mathscr{D}_n + (\varphi_n)_{\mathscr{X}_n},$$

we have

$$((f^{\mathrm{an}})^n)^*(g_{(\mathscr{X}, \mathscr{D})}) = d^n g_{(\mathscr{X}_n, \mathscr{D}_n)} - \log|\varphi_n|^2,$$

so that if we set $\theta = g - g_{(\mathscr{X}, \mathscr{D})}$, then $((f^{\mathrm{an}})^n)^*(\theta) = d^n \theta_n$. Therefore,

$$\|\theta\|_{\sup} = \|((f^{\mathrm{an}})^n)^*(\theta)\|_{\sup} = \|d^n \theta_n\|_{\sup} = d^n \|\theta_n\|_{\sup},$$

and hence $\lim_{n \to \infty} \|\theta_n\|_{\sup} = 0$.

For the last statement, note that if $\mathscr{D}$ is relatively nef, then $\mathscr{D}_n$ is also relatively nef for $n \geq 1$. $\qquad\qquad\blacksquare$

3.2   Complex case

We assume that $K = \mathbb{C}$.

Proposition 3.5. There exists a unique D-Green function g of C^0-type on X such that $f^*(g) = dg - \log|\varphi|^2$. Moreover, if there is a D-Green function of $(C^0 \cap \mathrm{PSH})$-type, then g is also of $(C^0 \cap \mathrm{PSH})$-type. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proof.   We can prove the unique existence of g in the same way as Proposition 3.1. Let g_0 be a D-Green function of $(C^0 \cap \mathrm{PSH})$-type. As in the previous subsection, we can see

$$(f^n)^*(D) = d^n D + (\varphi_n) \quad \text{and} \quad (f^n)^*(g) = d^n g - \log|\varphi_n|^2$$

for $n \geq 1$, where

$$\varphi_n := \prod_{i=0}^{n-1} ((f^{n-1-i})^*(\varphi))^{d^i}.$$

Here we define g_n to be

$$g_n := \frac{1}{d^n}((f^n)^*(g_0) + \log|\varphi_n|^2),$$

that is,

$$(f^n)^*(g_0) = d^n g_n - \log|\varphi_n|^2.$$

Then g_n is a D-Green function of $(C^0 \cap \mathrm{PSH})$-type. Moreover, if we set $\theta = g - g_0$ and $\theta_n = g - g_n$, then $(f^n)^*(\theta) = d^n \theta_n$. Thus,

$$\|\theta\|_{\sup} = \|(f^n)^*(\theta)\|_{\sup} = \|d^n \theta_n\|_{\sup} = d^n \|\theta_n\|_{\sup},$$

and hence $\lim_{n\to\infty} \|\theta_n\|_{\sup} = 0$. Therefore, g is of $(C^0 \cap \mathrm{PSH})$-type by [13, Theorem 2.9.14, (iii)]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \blacksquare$

Let $c: \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ be the morphism given by the complex conjugation map $z \mapsto \bar{z}$. Let $\tilde{X}$ denote the fiber product $X \times_{\mathrm{Spec}(\mathbb{C})}^c \mathrm{Spec}(\mathbb{C})$ in terms of c. Let $F: \tilde{X} \to X$ be the projection morphism and $\tilde{f}: \tilde{X} \to \tilde{X}$ be the induced morphism by f. Note that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ X & \xrightarrow{f} & X \end{array}$$

If we set $\tilde{D} = F^*(D)$ and $\tilde{\varphi} = F^*(\varphi)$, then $\tilde{f}^*(\tilde{D}) = d\tilde{D} + (\tilde{\varphi})$. For $x \in \tilde{X}(\mathbb{C})$, the composition $\mathrm{Spec}(\mathbb{C}) \xrightarrow{c} \mathrm{Spec}(\mathbb{C}) \xrightarrow{x} \tilde{X} \xrightarrow{F} X$ yields a $\mathbb{C}$-valued point of X, so that we define

$F_\infty : \tilde{X}(\mathbb{C}) \to X(\mathbb{C})$ to be $F_\infty(x) = F \circ x \circ c$. The above commutative diagram gives rise to the following commutative diagram:

$$\begin{array}{ccc}
\tilde{X}(\mathbb{C}) & \overset{\tilde{f}}{\longrightarrow} & \tilde{X}(\mathbb{C}) \\
F_\infty \downarrow & & \downarrow F_\infty \\
X(\mathbb{C}) & \underset{f}{\longrightarrow} & X(\mathbb{C})
\end{array}$$

Proposition 3.6. Let g be a D-Green function of C^0-type on X with $f^*(g) = dg - \log|\varphi|^2$ as in Proposition 3.5. Then $\tilde{g} := F_\infty^*(g)$ is a $\tilde{D}$-Green function of C^0-type on $\tilde{X}$ with $\tilde{f}^*(\tilde{g}) = d\tilde{g} - \log|\tilde{\varphi}|^2$. Moreover, if g is of $(C^0 \cap \mathrm{PSH})$-type, then $\tilde{g}$ is also of $(C^0 \cap \mathrm{PSH})$-type. $\qquad\square$

Proof. It is easy to see that $\tilde{g}$ is a $\tilde{D}$-Green function of C^0-type on $\tilde{X}$ because

$$F^*(\psi)(x) = x^*(F^*(\psi)) = c^*(c^*(x^*(F^*(\psi)))) = \overline{(F_\infty(x))^*(\psi)} = \overline{\psi(F_\infty(x))}$$

for $x \in \tilde{X}(\mathbb{C})$ and $\psi \in \mathrm{Rat}(X)^\times$. In addition,

$$\tilde{f}^*(\tilde{g}) = F_\infty^*(f^*(g)) = F_\infty^*(dg - \log|\varphi|^2) = d\tilde{g} - \log|\tilde{\varphi}|^2.$$

The last assertion follows from the same argument of [15, Lemma 5.1.1]. $\qquad\blacksquare$

4 Canonical Compactification

Let X be a projective and geometrically integral variety over a number field K. Let $f : X \to X$ be a surjective endomorphism of X over K. Let D be an $\mathbb{R}$-Cartier divisor on X. We assume that there are a real number d and $\varphi \in \mathrm{Rat}(X)^\times_\mathbb{R}$ such that $d > 1$ and $f^*(D) = dD + (\varphi)$. We use the same notation as in Sections 1.1.1–1.1.4. In addition, for each $v \in M_K \cup K(\mathbb{C})$, let $f_v^{\mathrm{an}} : X_v^{\mathrm{an}} \to X_v^{\mathrm{an}}$ be the induced map by f. By Proposition 3.1, for $\mathfrak{p} \in M_K$, there is a unique D-Green function $g_\mathfrak{p}$ of C^0-type on $X_\mathfrak{p}^{\mathrm{an}}$ with

$$(f_\mathfrak{p}^{\mathrm{an}})^*(g_\mathfrak{p}) = dg_\mathfrak{p} - \log|\varphi|_\mathfrak{p}^2.$$

We can find a model $\mathscr{X}_U$ of X over a nonempty Zariski open set U of $\mathrm{Spec}(O_K)$, an $\mathbb{R}$-Cartier divisor $\mathscr{D}_U$ on $\mathscr{X}_U$, and an endomorphism $f_U : \mathscr{X}_U \to \mathscr{X}_U$ over U such that $f_U|_X = f$ and $f_U^*(\mathscr{D}) = d\mathscr{D} + (\varphi)$ on $\mathscr{X}_U$, so that, by Proposition 3.3, for $P \in U$, g_P comes from the model $(\mathscr{X}_U, \mathscr{D}_U)$. Further, by virtue of Propositions 3.5 and 3.6, let us take a

unique F_∞-invariant D-Green function g_∞ of C^0-type on X_∞^{an} (for the definition of X_∞^{an}, see Section 1.1.3) such that

$$(f_\infty^{\mathrm{an}})^*(g_\infty) = dg_\infty - \log |\varphi|_\infty^2,$$

where $f_\infty^{\mathrm{an}} := \coprod_{\sigma \in K(\mathbb{C})} f_\sigma^{\mathrm{an}}$. Therefore,

$$\bar{D} := \left(D, \sum_{P \in M_K} g_P[P] + g_\infty[\infty] \right)$$

forms an adelic arithmetic Cartier divisor of C^0-type on X. By our construction,

$$f^*(\bar{D}) = d\bar{D} + \widehat{(\varphi)}.$$

The adelic arithmetic Cartier divisor $\bar{D}$ is called *the canonical compactification of D* with respect to f.

Lemma 4.1. If D is ample, that is, there are ample Cartier divisors $D_1, \ldots, D_r$ on X and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ with $D = a_1 D_1 + \cdots + a_r D_r$, then $\bar{D}$ is nef. $\qquad\square$

Proof. First let us see the following claim.

Claim 4.2.

(a) There are a model $\pi : \mathscr{X} \to \mathrm{Spec}(O_K)$ of X over $\mathrm{Spec}(O_K)$ and a relatively nef $\mathbb{R}$-Cartier divisor $\mathscr{D}$ on $\mathscr{X}$ such that $\mathscr{D}|_X = D$.

(b) There is an F_∞-invariant D-Green function h of C^∞-type on X_∞^{an} such that $c_1(D, h)$ is positive. $\qquad\blacksquare$

Proof. If D is an ample Cartier divisor, then the assertions (a) and (b) are well known. Moreover, in this case, $\mathscr{D}$ in (a) can be taken as a $\mathbb{Q}$-Cartier divisor.

(a) For each $i = 1, \ldots, r$, there are a model $\mathscr{X}_i \to \mathrm{Spec}(O_K)$ of X over $\mathrm{Spec}(O_K)$ and a relatively nef $\mathbb{Q}$-Cartier divisor $\mathscr{D}_i$ on $\mathscr{X}_i$ such that $\mathscr{D}_i|_X = D_i$. Let us take a model $\mathscr{X} \to \mathrm{Spec}(O_K)$ of X over $\mathrm{Spec}(O_K)$ such that we have a birational morphism $\mu_i : \mathscr{X} \to \mathscr{X}_i$ over $\mathrm{Spec}(O_K)$ for each $i = 1, \ldots, r$. If we set $\mathscr{D} = a_1 \mu_1^*(\mathscr{D}_1) + \cdots + a_r \mu_r^*(\mathscr{D}_r)$, then $\mathscr{D}$ is relatively nef and $\mathscr{D}|_X = D$.

(b) For each $i = 1, \ldots, r$, let h_i be an F_∞-invariant D_i-Green function of C^∞-type on X_∞^{an} such that $c_1(D_i, h_i)$ is positive. Then $a_1 h_1 + \cdots + a_r h_r$ is our desired Green function. $\qquad\blacksquare$

By the above claim together with Propositions 3.4 and 3.5, g_P and g_∞ are of $(C^0 \cap \mathrm{PSH})$-type on X_P^{an} and X_∞^{an}, respectively. Therefore, $\bar{D}$ is relatively nef. Let $h_{\bar{D}}$ be the height function associated with $\bar{D}$. Then

$$h_{\bar{D}}(f(x)) = d h_{\bar{D}}(x)$$

for all $x \in X(\bar{K})$. Indeed,

$$h_{\bar{D}}(f(x)) = h_{f^*(\bar{D})}(x) = h_{d\bar{D}+\widehat{(\varphi)}}(x) = h_{d\bar{D}}(x) = d h_{\bar{D}}(x).$$

As D is ample, there is a constant C such that $h_{\bar{D}} \geq C$. In particular,

$$h_{\bar{D}}(x) = h_{\bar{D}}(f^n(x))/d^n \geq C/d^n$$

for all $n \geq 1$, and hence $h_{\bar{D}}(x) \geq 0$ for $x \in X(\bar{K})$. Therefore, $\bar{D}$ is nef.

For $v \in M_K \cup K(\mathbb{C})$, we set

$$\begin{cases} \mathrm{Prep}(f) := \{x \in X(\bar{K}) \mid f^n(x) = f^m(x) \text{ for some } 0 \leq n < m\}, \\ \mathrm{Prep}(f_v) := \{x \in X_v(\bar{K}_v) \mid f_v^n(x) = f_v^m(x) \text{ for some } 0 \leq n < m\}. \end{cases}$$

An element of $\mathrm{Prep}(f)$ (respectively, $\mathrm{Prep}(f_v)$) is called a *prepriodic point* of f (respectively, f_v). Moreover, for a subset T of $X_v(\bar{K}_v)$, T^{an} is defined by

$$T^{\mathrm{an}} := \{w^{\mathrm{an}} \mid w \in T\}$$

(for the definition of w^{an}, see Section 1.1.4). Let us see the following proposition.

Proposition 4.3. $\bigcup_{x \in \mathrm{Prep}(f)} O_v(x) = \mathrm{Prep}(f_v)$ (for the definition of $O_v(x)$, see Section 1.1.2). □

Proof. Clearly, $\bigcup_{x \in \mathrm{Prep}(f)} O_v(x) \subseteq \mathrm{Prep}(f_v)$. Conversely, we suppose that $x \in \mathrm{Prep}(f_v)$, that is, $f_v^m \circ x = f_v^n \circ x$ for some $m > n \geq 0$. Let $\pi_v : X_v \to X$ be the projection. Then $\pi_v \circ f_v^m \circ x = \pi_v \circ f_v^n \circ x$. Note that the following diagram is commutative:

$$\begin{array}{ccc} X_v & \xrightarrow{\ f_v\ } & X_v \\ {\scriptstyle \pi_v}\downarrow & & \downarrow{\scriptstyle \pi_v} \\ X & \xrightarrow[\ f\ ]{} & X \end{array}$$

so that we have $f^m \circ \pi_v \circ x = f^n \circ \pi_v \circ x$. Therefore, by Lemma 4.4, there are a closed point ξ of X and a homomorphism $K(\xi) \to \bar{K}_v$ such that $\pi_v \circ x$ is given by the composition $\mathrm{Spec}(\bar{K}_v) \to \mathrm{Spec}(K(\xi)) \to X$, so that the assertion follows. ■

Lemma 4.4. Let V be a projective variety over a field k. Let $f : V \to V$ and $g : V \to V$ be surjective endomorphisms of V and let D be an $\mathbb{R}$-Cartier divisor on V. We assume the following:

(1) D is ample, that is, there are ample Cartier divisors $D_1, \ldots, D_r$ on V and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ with $D = a_1 D_1 + \cdots + a_r D_r$.

(2) There are $\phi, \psi \in \mathrm{Rat}(V)^\times_{\mathbb{R}}$ and $a, b \in \mathbb{R}_{>0}$ such that $f^*(D) = aD + (\phi)$, $g^*(D) = bD + (\psi)$ and $a \neq b$.

If Ω is a field over k, $x \in V(\Omega)$ and $f(x) = g(x)$, then there are a closed point ξ of V and a homomorphism $k(\xi) \to \Omega$ such that x coincides with the composition of $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(k(\xi)) \to V$. ☐

Proof. Let Z denote $(f \times g)^{-1}(\Delta)$, where $f \times g : V \to V \times V$ is a morphism given by $x \mapsto (f(x), g(x))$ and Δ is the diagonal in $V \times V$. It is sufficient to show that $\dim Z \leq 0$. We assume the contrary, so that we can find a subvariety C of dimension one of V with $C \subseteq Z$. Then $f|_C = g|_C$. In particular, $f_*(C) = g_*(C)$. As

$$f^*(D) - g^*(D) = (a - b)D + (\phi\psi^{-1}),$$

we have

$$(a - b)(D \cdot C) = ((f^*(D) - g^*(D)) \cdot C) = (D \cdot f_*(C)) - (D \cdot g_*(C)) = 0,$$

and hence $(D \cdot C) = 0$. This is a contradiction because D is ample. ■

The purpose of this section is to prove the following theorem.

Theorem 4.5. We assume that D is ample. If there are $v \in M_K \cup K(\mathbb{C})$ and a subvariety $Y_v \subseteq X_v$ such that $\dim Y_v \geq 1$ and $Y_v \subseteq \mathrm{Supp}_{\mathrm{ess}}(\mathrm{Prep}(f))^{\mathrm{an}}_v$, then the Dirichlet property of $\bar{D}$ does not hold. In particular, if $\mathrm{Prep}(f_v)^{\mathrm{an}}$ is dense in X^{an}_v with respect to the analytic topology for some $v \in M_K \cup K(\mathbb{C})$, then the Dirichlet property of $\bar{D}$ does not hold. ☐

Proof.   We assume that $\bar{D} + \widehat{(s)}$ is effective for some $s \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$. Here we set $S := X(\bar{K})^{\bar{D}}_{\leq 0}$ (for the definition of $X(\bar{K})^{\bar{D}}_{\leq 0}$, see Section 6). By Lemma 2.1,

$$\mathrm{Supp}_{\mathrm{ess}}(S)^{\mathrm{an}}_v \cap \{x \in X^{\mathrm{an}}_v \mid |s|_{g_v}(x) < 1\} = \emptyset.$$

Note that if $x \in \mathrm{Prep}(f)$ for $x \in X(\bar{K})$, then $h_{\bar{D}}(x) = 0$. Therefore, $\mathrm{Prep}(f) \subseteq S$, and hence

$$\mathrm{Supp}_{\mathrm{ess}}(\mathrm{Prep}(f))^{\mathrm{an}}_v \cap \mathrm{Supp}(D + (s))^{\mathrm{an}}_v = \emptyset$$

because $\mathrm{Supp}(D + (s))^{\mathrm{an}}_v \subseteq \{x \in X^{\mathrm{an}}_v \mid |s|_{g_v}(x) < 1\}$. As $(D + (s))_v$ is ample, we can see that $Y_v \cap \mathrm{Supp}(D + (s))_v \neq \emptyset$. In particular,

$$Y^{\mathrm{an}}_v \cap \mathrm{Supp}(D + (s))^{\mathrm{an}}_v \neq \emptyset,$$

which is a contradiction because $Y^{\mathrm{an}}_v \subseteq \mathrm{Supp}_{\mathrm{ess}}(\mathrm{Prep}(f))^{\mathrm{an}}_v$.   ∎

5   Examples

In this section, we give several examples to apply Theorem 4.5.

Example 5.1 (Abelian variety).   Let A be an abelian variety over a number field K. Let D be an ample and symmetric $\mathbb{R}$-Cartier divisor on A, that is, there are ample and symmetric Cartier divisors $D_1, \ldots, D_r$ on A and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ with $D = a_1 D_1 + \cdots + a_r D_r$. Then $[2]^*(D) = 4D + (\varphi)$ for some $\varphi \in \mathrm{Rat}(A)^{\times}_{\mathbb{R}}$. Let $\bar{D}$ be the canonical compactification of D with respect to $[2]$. Note that $\mathrm{Prep}([2]_\sigma)$ is dense in $A_\sigma(\mathbb{C})$ with respect to the analytic topology for $\sigma \in K(\mathbb{C})$. Thus, by Lemma 4.1 and Theorem 4.5, $\bar{D}$ is nef and $\bar{D}$ does not have the Dirichlet property.   □

Example 5.2 (Lattès map).   Let E be an elliptic curve over a number field K, $X := E/[\pm 1]$ and $\rho : E \to X$ be the natural morphism. Note that $X \simeq \mathbb{P}^1_K$ and the endomorphism $[2] : E \to E$ descends to an endomorphism $f : X \to X$, that is, the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{[2]} & E \\ \rho \downarrow & & \downarrow \rho \\ X & \xrightarrow{f} & X \end{array}$$

Clearly, $\rho(\mathrm{Prep}([2])) \subseteq \mathrm{Prep}(f)$. In particular, $\mathrm{Prep}([f]_\sigma)$ is dense in $X_\sigma(\mathbb{C})$ with respect to the analytic topology for $\sigma \in K(\mathbb{C})$. Let D be an ample Cartier divisor on X. Note that

$\rho^*(D)$ is symmetric because $\rho \circ [-1] = \rho$, so that there is $\varphi' \in \mathrm{Rat}(E)^\times$ with $[2]^*(\rho^*(D)) = 4\rho^*(D) + (\varphi')$, that is, $\rho^*(f^*(D) - 4D) = (\varphi')$. Therefore, if we set $\varphi = N(\varphi')^{1/2} \in \mathrm{Rat}(X)_{\mathbb{Q}}^\times$, then $f^*(D) = 4D + (\varphi)$, where $N : \mathrm{Rat}(E)^\times \to \mathrm{Rat}(X)^\times$ is the norm map. Let $\bar{D}$ be the canonical compactification of D with respect to f. By Lemma 4.1 and Theorem 4.5, $\bar{D}$ is nef and the Dirichlet property of $\bar{D}$ does not hold.

Here let us consider a special elliptic curve E due to Tate, that is,

$$E := \mathrm{Proj}(K[X, Y, Z]/(Y^2 Z + XYZ + \epsilon^2 Y Z^2 - X^3)),$$

where $\epsilon = (5 + \sqrt{29})/2$ and $K = \mathbb{Q}(\epsilon)$. It has a smooth model

$$\mathscr{E} = \mathrm{Proj}(O_K[X, Y, Z]/(Y^2 Z + XYZ + \epsilon^2 Y Z^2 - X^3))$$

over $O_K := \mathbb{Z}[\epsilon]$. Let $\mathscr{E} \dashrightarrow \mathbb{P}^1_{O_K}$ be a rational map induced by the homomorphism $O_K[X, Z] \to O_K[X, Y, Z]/(Y^2 Z + XYZ + \epsilon^2 Y Z^2 - X^3)$, that is, $\mathscr{E} \dashrightarrow \mathbb{P}^1_{O_K}$ is the projection at $(0 : 1 : 0)$. Note that $\mathscr{E} \dashrightarrow \mathbb{P}^1_{O_K}$ actually extends to a morphism $\rho : \mathscr{E} \to \mathbb{P}^1_{O_K}$ because the tangent line at $(0 : 1 : 0)$ is given by $\{Z = 0\}$.

Claim 5.3. There is a morphism $f : \mathbb{P}^1_{O_K} \to \mathbb{P}^1_{O_K}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathscr{E} & \xrightarrow{\ [2]\ } & \mathscr{E} \\
{\scriptstyle \rho} \downarrow & & \downarrow {\scriptstyle \rho} \\
\mathbb{P}^1_{O_K} & \xrightarrow{\ f\ } & \mathbb{P}^1_{O_K}
\end{array}
$$
□

Proof.   The x-coordinate of $[2](P)$ for $P = (x : y : 1) \in E$ is given by

$$\frac{x^4 - \epsilon^2 x^2 - 2\epsilon^4 x}{4x^3 + x^2 + 2\epsilon^2 x + \epsilon^4}.$$

Therefore, if we consider a rational map $f : \mathbb{P}^1_{O_K} \dashrightarrow \mathbb{P}^1_{O_K}$ given by

$$f(x : z) := (x^4 - \epsilon^2 x^2 z^2 - 2\epsilon^4 x z^3 : 4x^3 z + x^2 z^2 + 2\epsilon^2 x z^3 + \epsilon^4 z^4),$$

then the diagram

$$
\begin{array}{ccc}
\mathscr{E} & \xrightarrow{\;[2]\;} & \mathscr{E} \\
{\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\
\mathbb{P}^1_{O_K} & \dashrightarrow[\;f\;] & \mathbb{P}^1_{O_K}
\end{array}
$$

is commutative as rational maps, so that we need to see that f extends to a morphism $f : \mathbb{P}^1_{O_K} \to \mathbb{P}^1_{O_K}$. Let F be either $\bar{\mathbb{Q}}$ or $\bar{\mathbb{F}}_p$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime p. It is sufficient to show that if $(x, z) \in F^2$ satisfies a system of equations

$$
\begin{cases}
x^4 - \epsilon^2 x^2 z^2 - 2\epsilon^4 x z^3 = 0, \\
4x^3 z + x^2 z^2 + 2\epsilon^2 x z^3 + \epsilon^4 z^4 = 0,
\end{cases}
$$

then $x = z = 0$. We assume the contrary, that is, the above has a solution $(x, z) \in F^2 \setminus \{(0, 0)\}$. As $z \neq 0$, we may assume $z = 1$, so that $x \neq 0$, and hence

$$
\begin{cases}
x^3 - \epsilon^2 x - 2\epsilon^4 = 0, \\
4x^3 + x^2 + 2\epsilon^2 x + \epsilon^4 = 0.
\end{cases}
$$

Therefore, $0 = (4x^3 + x^2 + 2\epsilon^2 x + \epsilon^4) - 4(x^3 - \epsilon^2 x - 2\epsilon^4) = (x + 3\epsilon^2)^2$, that is, $x = -3\epsilon^2$. Thus, as $(-3\epsilon^2)^3 - \epsilon^2(-3\epsilon^2) - 2\epsilon^4 = 0$ and $\epsilon \neq 0$, we have $27\epsilon^2 = 1$. On the other hand, since $\epsilon^2 - 5\epsilon - 1 = 0$, we obtain $27 \cdot 5\epsilon = -26$, so that $27 \cdot 25 = 27 \cdot 25 \cdot 27\epsilon^2 = (27 \cdot 5\epsilon)^2 = 26^2$ in F. Note that $26^2 - 27 \cdot 25 = 1$, and hence $1 = 0$ in F, which is a contradiction. ∎

Since the norm map $N : \mathrm{Rat}(\mathscr{E})^\times \to \mathrm{Rat}(\mathbb{P}^1_{O_K})^\times$ is a homomorphism, we have the natural extension

$$
N_{\mathbb{Q}} : \mathrm{Rat}(\mathscr{E})^\times_{\mathbb{Q}} := \mathrm{Rat}(\mathscr{E})^\times \otimes \mathbb{Q} \longrightarrow \mathrm{Rat}(\mathbb{P}^1_{O_K})^\times_{\mathbb{Q}} := \mathrm{Rat}(\mathbb{P}^1_{O_K})^\times \otimes \mathbb{Q}.
$$

Let $\mathscr{D}$ be an ample Cartier divisor on $\mathbb{P}^1_{O_K}$. As the following diagram:

$$
\begin{array}{ccc}
\mathscr{E} & \xrightarrow{\;[-1]\;} & \mathscr{E} \\
& {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\rho} & \\
& \mathbb{P}^1_{O_K} &
\end{array}
$$

is commutative, $\rho^*(\mathscr{D})$ is symmetric, so that $[2]^*(\rho^*(\mathscr{D})) - 4\rho^*(\mathscr{D}) - (\varphi')$ is vertical for some $\varphi' \in \mathrm{Rat}(\mathscr{E})^\times$. As the class group of $\mathbb{Q}(\epsilon)$ is finite, there is $\lambda \in K^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\rho^*(f^*(\mathscr{D}) - 4\mathscr{D}) = [2]^*(\rho^*(\mathscr{D})) - 4\rho^*(\mathscr{D}) = (\lambda\varphi'),$$

and hence, if we set $\varphi = N_{\mathbb{Q}}(\lambda\varphi')^{1/2} \in \mathrm{Rat}(\mathbb{P}^1_{O_K})^\times_{\mathbb{Q}}$, then $f^*(\mathscr{D}) = 4\mathscr{D} + (\varphi)$. Let g_∞ be an F_∞-invariant $\mathscr{D}$-Green function of $(C^0 \cap \mathrm{PSH})$-type on $\mathbb{P}^1_{O_K}(\mathbb{C})$ such that $f^*(g) = 4g - \log|\varphi|^2_\infty$. By Lemma 4.1 and [17, Proposition 2.1.7], the arithmetic Cartier divisor $\bar{\mathscr{D}} := (\mathscr{D}, g_\infty)$ on $\mathbb{P}^1_{O_K}$ is nef and, by Theorem 4.5, $\bar{\mathscr{D}} + \widehat{(\psi)}$ is not effective for all $\psi \in \mathrm{Rat}(\mathbb{P}^1_{O_K})^\times_{\mathbb{R}}$. Further, $\rho^*(\bar{\mathscr{D}})$ is nef and $\rho^*(\bar{\mathscr{D}}) + \widehat{(\phi)}$ is not effective for all $\phi \in \mathrm{Rat}(\mathscr{A})^\times_{\mathbb{R}}$. $\square$

Example 5.4.   Here let us give an example due to Burgos i Gil, which shows that the converse of Theorem 1.1 in Section 1 does not hold in general.

Let E be an elliptic curve over $\mathbb{Q}$ and $\mathbb{P}^1_{\mathbb{Q}} := \mathrm{Proj}(\mathbb{Q}[x, y])$. Let D_1 (respectively, D_2) be the Cartier divisor on E (respectively, $\mathbb{P}^1_{\mathbb{Q}}$) given by the zero point (respectively, $\{x = 0\}$). Then there is $\varphi \in \mathrm{Rat}(E)^\times$ with $[2]^*(D_1) = 4D_1 + (\varphi)$. Let $h : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be the endomorphism given by $(x : y) \mapsto (x^4 : y^4)$. Then $h^*(D_2) = 4D_2$. We set

$$X := E \times \mathbb{P}^1_{\mathbb{Q}}, \quad f := [2] \times h : X \to X, \quad \text{and} \quad D := p_1^*(D_1) + p_2^*(D_2),$$

where $p_1 : X \to E$ and $p_2 : X \to \mathbb{P}^1_{\mathbb{Q}}$ are the projections to E and $\mathbb{P}^1_{\mathbb{Q}}$, respectively. As the following diagrams are commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
E & \xrightarrow[{[2]}]{} & E
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\
\mathbb{P}^1_{\mathbb{Q}} & \xrightarrow[h]{} & \mathbb{P}^1_{\mathbb{Q}}
\end{array}
$$

we have

$$f^*(D) = f^*(p_1^*(D_1)) + f^*(p_2^*(D_2)) = p_1^*([2]^*(D_1)) + p_2^*(h^*(D_2))$$

$$= p_1^*(4D_1 + (\varphi)) + p_2^*(4D_2) = 4D + (p_1^*(\varphi)).$$

Let $\bar{D}$ be the canonical compactification of D with respect to f.

Claim 5.5.

    (1) $\bar{D}$ is nef.

    (2) $\mathrm{Prep}(f_v)^{\mathrm{an}}$ is not dense in X_v^{an} with respect to the analytic topology for all $v \in M_{\mathbb{Q}} \cup \{\infty\}$, where ∞ is the unique embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$.

    (3) $\mathrm{Supp}_{\mathrm{ess}}(\mathrm{Prep}(f))_{\infty}^{\mathrm{an}} = E(\mathbb{C}) \times S^1$, where $S^1 = \{(\zeta:1) \in \mathbb{P}^1(\mathbb{C}) \mid |\zeta| = 1\}$.

    (4)   The Dirichlet property of $\bar{D}$ does not hold. $\square$

Proof.   Part (1) follows from Lemma 4.1.

    (2) As $(p_2)_v^{\mathrm{an}}: X_v^{\mathrm{an}} \to (\mathbb{P}_{\mathbb{Q}}^1)_v^{\mathrm{an}}$ is surjective and $(p_2)_v^{\mathrm{an}}(\mathrm{Prep}(f_v)^{\mathrm{an}}) \subseteq \mathrm{Prep}(h_v)^{\mathrm{an}}$, it is sufficient to show that $\mathrm{Prep}(h_v)^{\mathrm{an}}$ is not dense in $(\mathbb{P}_{\mathbb{Q}}^1)_v^{\mathrm{an}}$. Note that

$$\mathrm{Prep}(h) = \{(0:1), (1:0)\} \cup \{(\zeta, 1) \mid \zeta \in \bar{\mathbb{Q}} \text{ and } \zeta^m = 1 \text{ for some } m \in \mathbb{Z}_{>0}\},$$

so that the assertion is obvious if $v = \infty$. We assume that $v = p$ for some prime p. Let $w \in \mathrm{Prep}(h_p)^{\mathrm{an}} \cap U_{\mathbb{Q}_p}^{\mathrm{an}}$, where U is the Zariski open set of $\mathbb{P}_{\mathbb{Q}}^1$ given by $U := \{x \neq 0, \ y \neq 0\}$ and $U_{\mathbb{Q}_p} := U \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{Q}_p)$. In the same way as Proposition 4.3, there is $\xi \in \mathrm{Prep}(h)$ such that w is one of valuations arising from ξ, that is, if we set

$$\mathbb{Q}(\xi) \otimes \mathbb{Q}_p = K_1 \oplus \cdots \oplus K_r \quad \text{(the sum of finite extension fields over } \mathbb{Q}_p),$$

then w is the valuation of some K_i. Put $z := X/Y$. As $z(\xi)^m = 1$ for some $m \in \mathbb{Z}_{>0}$, we obtain $z^m = 1$ at K_i, so that $|z|_w = 1$. Therefore, we have

$$\mathrm{Prep}(h_p)^{\mathrm{an}} \cap U_{\mathbb{Q}_p}^{\mathrm{an}} \subseteq \{w \in U_{\mathbb{Q}_p}^{\mathrm{an}} \mid |z|_w = 1\},$$

and hence $\mathrm{Prep}(h_p)^{\mathrm{an}}$ is not dense.

    (3) We need to see $\overline{\mathrm{Prep}(f) \setminus Y(\mathbb{C})} = E(\mathbb{C}) \times S^1$ for any closed subscheme Y of $E \times \mathbb{P}_{\mathbb{Q}}^1$ which is strictly contained in $E \times \mathbb{P}_{\mathbb{Q}}^1$. Note that

$$\overline{\mathrm{Prep}(f)} = E(\mathbb{C}) \times S^1 \quad \text{and} \quad \overline{\mathrm{Prep}(f)} \setminus Y(\mathbb{C}) \subseteq \overline{\mathrm{Prep}(f) \setminus Y(\mathbb{C})},$$

so that it is sufficient to check $E(\mathbb{C}) \times S^1 \subseteq \overline{(E(\mathbb{C}) \times S^1) \setminus Y(\mathbb{C})}$.

    We set $T = \{\zeta \in S^1 \mid E(\mathbb{C}) \times \{\zeta\} \subseteq Y(\mathbb{C})\}$. Let us see that T is finite. Otherwise, as $E(\mathbb{C}) \times T \subseteq Y(\mathbb{C})$ and $E(\mathbb{C}) \times T$ is Zariski dense in $E(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, we have $Y(\mathbb{C}) = E(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, which is a contradiction.

    Since $\overline{(E(\mathbb{C}) \times \{\zeta\}) \setminus Y(\mathbb{C})} = E(\mathbb{C}) \times \{\zeta\}$ for $\zeta \in S^1 \setminus T$, we obtain

$$E(\mathbb{C}) \times (S^1 \setminus T) \subseteq \overline{E(\mathbb{C}) \times S^1 \setminus Y(\mathbb{C})}.$$

Thus, the assertion follows because $\overline{S^1 \setminus T} = S^1$.

    Part (4) follows from (3) and Theorem 4.5.   ■

6 Functional Approach to the Study of Dirichlet Property

The purpose of the section is to study the Dirichlet property in a functional point of view. Our method consists of introducing some (possible nonlinear) functionals on the spaces of continuous functions on the analytic fibers of the arithmetic variety. The Dirichlet property leads to conditions on the supports of these functionals.

We fix a geometrically integral projective scheme X of dimension d defined over a number field K and denote by $\pi : X \to \operatorname{Spec} K$ the structural morphism. In the following subsection, we will establish an abstract framework to study the consequences of the Dirichlet property by the functional approach. We then specify the theorem for different choices of the functionals, notably those coming from the asymptotic maximal slope and the volume function.

6.1 A formal functional analysis on Dirichlet property

Let V be a vector subspace of $\widehat{\operatorname{Div}}_{\mathbb{R}}(X)$ containing all principal divisors and let V_+ denote the subset of all effective adelic arithmetic Cartier divisors in V. Let $C_\circ$ be a subset of V verifying the following conditions:

(a)   for any $\bar{D} \in C_\circ$ and $\lambda > 0$, one has $\lambda \bar{D} \in C_\circ$;

(b)   for any $\bar{D}_0 \in C_\circ$ and $\bar{D} \in V_+$, there exists $\varepsilon_0 > 0$ such that $\bar{D}_0 + \varepsilon \bar{D} \in C_\circ$ for any $\varepsilon \in \mathbb{R}$ with $0 \le \varepsilon \le \varepsilon_0$;

(c)   for any $\bar{D} \in C_\circ$ and $\phi \in \operatorname{Rat}(X)_{\mathbb{R}}^{\times}$, one has $\bar{D} + \widehat{(\phi)} \in C_\circ$.

In other terms, $C_\circ$ is a cone in V which is open in the directions in V_+ and invariant under translations by a principal divisor.

Let $\mu : C_\circ \to \mathbb{R}$ be a map which verifies the following properties:

(1)   there exists a positive number a such that $\mu(t\bar{D}) = t^a \mu(\bar{D})$ for all adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D} \in C_\circ$ and $t > 0$;

(2)   for any $\bar{D} \in C_\circ$ and $\phi \in \operatorname{Rat}(X)_{\mathbb{R}}^{\times}$, one has $\mu(\bar{D} + \widehat{(\phi)}) = \mu(\bar{D})$.

For $\bar{D} \in C_\circ$ and $\bar{E} \in V_+$, we define $\nabla_{\bar{E}}^{+} \mu(\bar{D})$ to be

$$\nabla_{\bar{E}}^{+} \mu(\bar{D}) = \limsup_{\epsilon \to 0+} \frac{\mu(\bar{D} + \epsilon \bar{E}) - \mu(\bar{D})}{\epsilon},$$

which might be $\pm\infty$. Note that, for any $\bar{D} \in C_\circ$, the function $\bar{E} \mapsto \nabla_{\bar{E}}^{+} \mu(\bar{D})$ is positively homogeneous, namely $\nabla_{t\bar{E}}^{+} \mu(\bar{D}) = t \nabla_{\bar{E}}^{+} \mu(\bar{D})$ for any $\bar{E}$ and any $t > 0$. Moreover, for any

$\bar{E} \in V$ and $t > 0$, one has $\nabla^+_{\bar{E}} \mu(t\bar{D}) = t^{a-1} \nabla^+_{\bar{E}}(\bar{D})$. In addition to (1) and (2), assume the following property:

(3)   there exists a map $\nabla_\mu : \widehat{\mathrm{Div}}_{\mathbb{R}}(X)_+ \times C_\circ \to \mathbb{R} \cup \{\pm\infty\}$ such that

$$\nabla_\mu(\bar{E}, \bar{D}) = \nabla^+_{\bar{E}} \mu(\bar{D}) \quad \text{for } \bar{E} \in V_+ \text{ and } \bar{D} \in C_\circ,$$

where $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)_+$ denotes the set of all effective adelic arithmetic $\mathbb{R}$-Cartier divisors.

Denote by $C_{\circ\circ}$ the subset of $C_\circ$ of adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D}$ such that, the map $\bar{E} \mapsto \nabla_\mu(\bar{E}, \bar{D})$ preserves the order, namely, for any couple $(\bar{E}_1, \bar{E}_2)$ of elements in $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)_+$ such that $\bar{E}_1 \le \bar{E}_2$, one has $\nabla_\mu(\bar{E}_1, \bar{D}) \le \nabla_\mu(\bar{E}_2, \bar{D})$. If $\bar{D}$ is an element in $C_{\circ\circ}$, then the map ∇_μ defines, for any $v \in M_K \cup K(\mathbb{C})$, a nonnecessarily additive functional

$$\Psi^\mu_{\bar{D},v} : C^0(X^{\mathrm{an}}_v)_+ \longrightarrow [0, +\infty], \quad \Psi^\mu_{\bar{D},v}(f_v) := \nabla_\mu(\bar{O}(f_v), \bar{D}),$$

where $C^0(X^{\mathrm{an}}_v)_+$ denotes the cone of nonnegative continuous functions on X^{an}_v.

Definition 6.1. We define the *support* of $\Psi^\mu_{\bar{D},v}$ to be the set $\mathrm{Supp}(\Psi^\mu_{\bar{D},v})$ of all $x \in X^{\mathrm{an}}_v$ such that $\Psi^\mu_{\bar{D},v}(f_v) > 0$ for any nonnegative continuous function f_v on X^{an}_v verifying $f_v(x) > 0$. This is a closed subset of X^{an}_v. In fact, if x is a point of X^{an}_v which is outside $\mathrm{Supp}(\Psi^\mu_{\bar{D},v})$, then there exists a nonnegative continuous function f_v on X^{an}_v such that $\Psi^\mu_{\bar{D},v}(f_v) = 0$ but $f_v(x) > 0$. Therefore, $\{y \in X^{\mathrm{an}}_v \mid f_v(y) > 0\}$ is an open neighborhood of x which is contained in $X^{\mathrm{an}}_v \setminus \mathrm{Supp}(\Psi^\mu_{\bar{D},v})$. Note that $F_\infty(\mathrm{Supp}(\Psi^\mu_{\bar{D},\sigma})) = \mathrm{Supp}(\Psi^\mu_{\bar{D},\bar{\sigma}})$ for $\sigma \in K(\mathbb{C})$ because $\Psi^\mu_{\bar{D},\sigma}(f_\sigma) = \Psi^\mu_{\bar{D},\bar{\sigma}}(F^*_\infty(f_\sigma))$ for $f_\sigma \in C^0(X^{\mathrm{an}}_\sigma)_+$. $\square$

Theorem 6.2. Let $\bar{D}$ be an element of $C_{\circ\circ}$ with $\mu(\bar{D}) = 0$. If s is an element of $\mathrm{Rat}(X)^\times_{\mathbb{R}}$ with $\bar{D} + \widehat{(s)} \ge 0$, then

$$\mathrm{Supp}(\Psi^\mu_{\bar{D},v}) \cap \{x \in X^{\mathrm{an}}_v \mid |s|_{g_v}(x) < 1\} = \varnothing$$

for any $v \in M_K \cup K(\mathbb{C})$. $\square$

Proof.   We set $\bar{D}' = \bar{D} + \widehat{(s)} = (D', g')$ and $f_v = \min\{g'_v, 1\}$. Thus, as

$$0 \le \bar{O}(f_v) \le \bar{D}'$$

and $\bar{D} \in C_{\circ\circ}$, one has

$$0 = \nabla_\mu((0, 0), \bar{D}) \le \Psi^\mu_{\bar{D},v}(f_v) = \nabla_\mu(\bar{O}(f_v), \bar{D}) \le \nabla_\mu(\bar{D}', \bar{D}) = \nabla^+_{\bar{D}'} \mu(\bar{D}).$$

On the other hand, by using the properties (1) and (2), one obtains

$$\mu(\bar{D} + \epsilon \bar{D}') - \mu(\bar{D}) = \mu(\bar{D} + \epsilon \bar{D}) - \mu(\bar{D}) = ((1 + \epsilon)^a - 1)\mu(\bar{D}),$$

and hence $\nabla^+_{\bar{D}'} \mu(\bar{D}) = a\mu(\bar{D}) = 0$. Therefore, $\Psi_{\bar{D},v}(f_v) = 0$, so that

$$\mathrm{Supp}(\Psi_{\bar{D},v}) \cap \{x \in X_v^{\mathrm{an}} \mid f_v(x) > 0\} = \varnothing.$$

Note that $g_v' = -\log |s|_{g_v}^2$. Thus, we can see that

$$\{x \in X_v^{\mathrm{an}} \mid f_v(x) > 0\} = \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\},$$

as required. ∎

Under the assumptions of Theorem 6.2, we have the following corollaries.

Corollary 6.3. Assume that the Dirichlet property holds for $\bar{D}$ and that D is big. For any $v \in M_K \cup K(\mathbb{C})$, if Z is a closed subvariety of X_v of dimension ≥ 1 such that $Z^{\mathrm{an}} \subseteq \mathrm{Supp}(\Psi_{\bar{D},v}^\mu)$, then Z is contained in the augmented base locus (see Section 1.1.9) of D_v. □

Proof. Let s be an element of $\mathrm{Rat}(X)_{\mathbb{R}}^\times$ with $\bar{D} + \widehat{(s)} \geq 0$. We introduce $\bar{D}' = \bar{D} + \widehat{(s)}$ as in the proof of the theorem. Assume that Z is a closed subvariety of X_v which is not contained in the augmented base locus of D_v (which identifies with the augmented base locus of D_v'). Then the restriction of D_v' on Z is a big $\mathbb{R}$-Cartier divisor since the restricted volume of D_v' on Z is > 0 (cf. [10]). Hence, $[D_v']$ has nonempty intersection with Y_v, which implies that $[D_v']^{\mathrm{an}} \cap Z^{\mathrm{an}} \neq \varnothing$. Therefore, by the previous theorem, Z^{an} cannot be contained in the support of the functional $\Psi_{\bar{D},v}^\mu$. ∎

Corollary 6.4. Assume that $(D \cdot C) > 0$ for any curve C on X. If the Dirichlet property holds for $\bar{D}$, then, for any $v \in M_K \cup K(\mathbb{C})$, there is no subvariety Z of X_v such that $\dim Z \geq 1$ and $Z^{\mathrm{an}} \subseteq \mathrm{Supp}(\Psi_{\bar{D},v}^\mu)$. □

Proof. It is sufficient to prove that $(D_v \cdot C) > 0$ for any curve C on X_v. Indeed, there are a variety W over K and a subscheme $\mathscr{C}$ of $X \times W$ such that $\mathrm{Rat}(W)$ is a subfield of K_v, $\mathscr{C}$ is flat over W and $\mathscr{C} \times_W \mathrm{Spec}(K_v) = C$. Let p and q be the projections $X \times W \to W$ and $X \times W \to X$, respectively. By our assumption, $(q^*(D) \cdot \mathscr{C} \cap p^{-1}(w)) > 0$ for any closed point w of W, so that $(q^*(D) \cdot \mathscr{C}_\eta) > 0$, where η is the generic point of W, as required. ∎

6.2 **Asymptotic maximal slope**

In this subsection, let $V = \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ and $C_\circ$ be the cone of all adelic arithmetic $\mathbb{R}$-Cartier divisors $\bar{D}$ such that D is big.

Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor in $C_\circ$ and ζ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}K$ with $\widehat{\deg}(\zeta) = 1$, we define $\hat{\mu}_{\max}^{\mathrm{asy},\zeta}(\bar{D})$ as

$$\sup\{t \in \mathbb{R} \mid \bar{D} - t\pi^*(\zeta) \text{ has the Dirichlet property}\}.$$

Note that for sufficiently negative number t, the adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D} - t\pi^*(\zeta)$ is big (since D is a big $\mathbb{R}$-divisor) and therefore has the Dirichlet property. Moreover, one has $\hat{\mu}_{\max}^{\mathrm{asy},\zeta}(\bar{D}) \leq \hat{\mu}_{\mathrm{ess}}(\bar{D})$ (see Section 1.1.6). Therefore, $\hat{\mu}_{\max}^{\mathrm{asy},\zeta}(.)$ is a real-valued function on $C_\circ$. By definition, for any $t \geq 0$ and any $\bar{D} \in C_\circ$ one has $\hat{\mu}_{\max}^{\mathrm{asy},\zeta}(t\bar{D}) = t\hat{\mu}_{\max}^{\mathrm{asy},\zeta}(\bar{D})$.

The function $\hat{\mu}_{\max}^{\mathrm{asy},\zeta}(\bar{D})$ is actually independent of the choice of ζ. This is a consequence of the following proposition.

Proposition 6.5. Let ζ_1 and ζ_2 be adelic arithmetic $\mathbb{R}$-Cartier divisors on $\mathrm{Spec}K$. Then $\widehat{\deg}(\zeta_1) = \widehat{\deg}(\zeta_2)$ if and only if $\zeta_1 = \zeta_2 + \widehat{(\varphi)}$ for some $\varphi \in K^\times \otimes_{\mathbb{Z}} \mathbb{R}$.   □

Proof. It is sufficient to show that if $\widehat{\deg}(\zeta) = 0$, then $\zeta = \widehat{(\varphi)}$ for some $\varphi \in K^\times \otimes_{\mathbb{Z}} \mathbb{R}$. We set $\zeta = \sum_{\mathfrak{p} \in M_K} a_{\mathfrak{p}}[\mathfrak{p}] + \sum_{\sigma \in K(\mathbb{C})} a_\sigma[\sigma]$. As the class group of K is finite, we may assume that $a_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in M_K$. Therefore, Dirichlet's unit theorem implies the assertion.   ∎

We shall use the expression $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ to denote this function. By definition, for any adelic arithmetic $\mathbb{R}$-Cartier divisor ζ one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + \pi^*(\zeta)) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) + \widehat{\deg}(\zeta).$$

This function has been introduced in the adelic line bundle setting in [8] in an equivalent form by using arithmetic graded linear series. We refer the readers to Section 7 *infra* for more details.

If $\bar{D}$ is an adelic arithmetic $\mathbb{R}$-Cartier divisor verifying the Dirichlet property, then for any $\varphi \in \mathrm{Rat}(X)_{\mathbb{R}}^\times$, $\bar{D} + \widehat{(\varphi)}$ also verifies the Dirichlet property. Moreover, for any $\bar{D}' \geq \bar{D}$, the adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D}'$ verifies the Dirichlet property. We deduce from these facts the following properties of the function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$.

Proposition 6.6. (1) Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor in $C_\circ$. For any $\varphi \in \mathrm{Rat}(X)_{\mathbb{R}}^\times$, one has $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + \widehat{(\varphi)}) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$.

(2) The function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ preserves the order relation, namely for $\bar{D}_1 \leq \bar{D}_2$ in $C_\circ$ one has $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1) \leq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_2)$.

(3) The function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ is super-additive, namely

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1 + \bar{D}_2) \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1) + \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_2)$$

for $\bar{D}_1$ and $\bar{D}_2$ in $C_\circ$. □

Theorem 6.2 leads immediately to the following corollary.

Corollary 6.7. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is big and that $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = 0$. If s is an element of $\mathrm{Rat}(X)_{\mathbb{R}}^\times$ with $\bar{D} + \widehat{(s)} \geq 0$, then

$$\mathrm{Supp}(\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}) \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} = \varnothing$$

for any $v \in M_K \cup K(\mathbb{C})$. □

The function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ is important in the study of Dirichlet's theorem. In fact, it is not only the threshold of the Dirichlet property but also the pseudo-effectivity.

Lemma 6.8. Let $(\bar{D}_i)_{i=1}^n$ be a family of adelic arithmetic $\mathbb{R}$-Cartier divisors on X, and $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X such that D is big. Then one has

$$\lim_{|\boldsymbol{t}| \to 0} \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + t_1 \bar{D}_1 + \cdots + t_n \bar{D}_n) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}),$$

where for $\boldsymbol{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$, the expression $|\boldsymbol{t}|$ denotes $\max\{|t_1|, \ldots, |t_n|\}$. □

Proof. If we replace $\bar{D}$ by $\bar{D} + \pi^*(\zeta)$, where ζ is an adelic arithmetic Cartier $\mathbb{R}$-divisor on $\mathrm{Spec}K$, both sides of the equality to be proved differ the initial value by $\widehat{\deg}(\zeta)$. Hence, one may assume that $\bar{D}$ is big. Moreover, without loss of generality, one may assume that $(\bar{D}_i)_{i=1}^n$ are adelic arithmetic Cartier divisors which are big and effective (by possibly augmenting the number of adelic arithmetic Cartier divisors in the family). In fact, each $\bar{D}_i$ is $\mathbb{R}$-linearly equivalent to an $\mathbb{R}$-linear combination of big and effective arithmetic Cartier divisors. Then by using the fact that the function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ preserves the order, one obtains

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} - |\boldsymbol{t}|(\bar{D}_1 + \cdots + \bar{D}_n)) \leq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + t_1 \bar{D}_1 + \cdots + t_n \bar{D}_n) \leq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + |\boldsymbol{t}|(\bar{D}_1 + \cdots + \bar{D}_n)).$$

Therefore, we have reduced the problem to the case where $n = 1$ and $\bar{D}_1$ is a big and effective adelic arithmetic Cartier divisor. Let $a > 0$ be a real number such that $a\bar{D} - \bar{D}_1$

and $a\bar{D} + \bar{D}_1$ are both $\mathbb{R}$-linearly equivalent to effective adelic arithmetic $\mathbb{R}$-Cartier divisors. By the positive homogenity of the function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$, for any $t > 0$ one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} - t\bar{D}_1) \geq (1 - at)\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}).$$

Hence, $\lim_{t \to 0+} \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} - t\bar{D}_1) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$. Similarly, one has $\lim_{t \to 0+} \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + t\bar{D}_1) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$. The result is thus proved. ∎

Proposition 6.9 (cf. Proposition 8.2)**.** Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is big. Then $\bar{D}$ is big (respectively, pseudo-effective) if and only if $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) > 0$ (respectively, $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq 0$). □

Proof. Assume that $\bar{D}$ is big. Let ζ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}\,K$ such that $\widehat{\deg}(\zeta) > 0$. Since $\bar{D}$ is big, for sufficiently small $t > 0$, the adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D} - t\pi^*(\zeta)$ is big, and hence verifies the Dirichlet property. Therefore, one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = t\widehat{\deg}(\zeta) + \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} - t\pi^*(\zeta)) \geq t\widehat{\deg}(\zeta) > 0.$$

Conversely, assume that $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) > 0$. We write $\bar{D}$ as an $\mathbb{R}$-linear combination

$$\bar{D} = a_1\bar{D}_1 + \cdots + a_n\bar{D}_n,$$

where $(\bar{D}_i)_{i=1}^n$ are big adelic arithmetic $\mathbb{Q}$-Cartier divisors. For any $\varepsilon > 0$, we can choose $b_1, \ldots, b_n$ in $\mathbb{Q}$ such that $a_i - \varepsilon \leq b_i < a_i$. Then $\bar{D}_\varepsilon = b_1\bar{D}_1 + \cdots + b_n\bar{D}_n$ is an adelic arithmetic $\mathbb{Q}$-Cartier divisor and $D - D_\varepsilon$ is big. Moreover, if ε is sufficiently small, D_ε is big and $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_\varepsilon) > 0$. By [7, Proposition 3.11] (see also Proposition 7.1), one obtains that $\bar{D}_\varepsilon$ is big. Therefore, $\bar{D}$ is also big.

Assume that $\bar{D}$ is pseudo-effective. Let $\bar{D}'$ be a big adelic arithmetic $\mathbb{R}$-Cartier divisor. For any $t > 0$, $t\bar{D}' + \bar{D}$ is big. Therefore, $\hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}' + \bar{D}) > 0$. By the continuity of the function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ (see Lemma 6.8), one obtains that $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq 0$.

Conversely, assume that $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq 0$. If $\bar{D}'$ is a big adelic arithmetic $\mathbb{R}$-Cartier divisor, then $D + D'$ is big since D is pseudo-effective and D' is big. Moreover, one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + \bar{D}') \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) + \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}') > 0.$$

Hence, $\bar{D} + \bar{D}'$ is big. Therefore, $\bar{D}$ is pseudo-effective. ∎

Remark 6.10. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X such that D is big. The above proposition shows that

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = \max\{t \in \mathbb{R} \mid \bar{D} - t\pi^*(\zeta) \text{ is pseudo-effective}\},$$

where ζ is any adelic arithmetic $\mathbb{R}$-Cartier divisor such that $\widehat{\deg}(\zeta) = 1$. Note that big adelic arithmetic $\mathbb{R}$-Cartier divisors verify the Dirichlet property. Therefore, in order to construct counter-examples to the Dirichlet property, one should examine adelic arithmetic $\mathbb{R}$-Cartier divisor of the form $\bar{D} - \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})\pi^*(\zeta)$, where $\widehat{\deg}(\zeta) = 1$. Note that the functionals $\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}$ remain invariant if one replaces $\bar{D}$ by a translation of $\bar{D}$ by the pull-back of an adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec} K$. Therefore, the study of these functionals will very possibly provide a large family of counter-examples to the Dirichlet property and suggest a way to characterize it. $\qquad\square$

6.3 Volume function

In this subsection, we still assume that $V = \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ and $C_\circ$ is the cone of all adelic arithmetic $\mathbb{R}$-Cartier divisors $\bar{D}$ such that D is big.

Let μ be the arithmetic volume function $\widehat{\mathrm{vol}}$ (see Section 1.1.8). Note that one has $\widehat{\mathrm{vol}}(t\bar{D}) = t^{d+1}\widehat{\mathrm{vol}}(\bar{D})$. Moreover, for any $\phi \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$, one has $\widehat{\mathrm{vol}}(\bar{D} + \widehat{(\phi)}) = \widehat{\mathrm{vol}}(\bar{D})$. Therefore, the function $\mu = \widehat{\mathrm{vol}}(.)$ verifies the conditions (1)–(3) in Section 6.1. Moreover, the volume function preserves the order relation. Namely for $\bar{D}_1 \leq \bar{D}_2$ one has $\widehat{\mathrm{vol}}(\bar{D}_1) \leq \widehat{\mathrm{vol}}(\bar{D}_2)$.

A direct consequence of Theorem 6.2 is the following corollary.

Corollary 6.11. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is big and that $\widehat{\mathrm{vol}}(\bar{D}) = 0$. If s is an element of $\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ with $\bar{D} + \widehat{(s)} \geq 0$, then

$$\mathrm{Supp}(\Psi_{\bar{D},v}^{\widehat{\mathrm{vol}}}) \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} = \varnothing$$

for any $v \in M_K \cup K(\mathbb{C})$. $\qquad\square$

6.4 Self-intersection number

In this subsection, let V be the subspace of $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ consisting of integrable adelic arithmetic $\mathbb{R}$-Cartier divisors. Let $C_\circ = V$. We define the function $\mu : C_\circ \to \mathbb{R}$ as $\mu(\bar{D}) := \widehat{\deg}(\bar{D}^{d+1})$. The function μ verifies the conditions (1) and (2) of Section 6.1. For $\bar{D} \in C_\circ$ and

$\bar{E} \in V_+$, we define $\nabla^+_{\bar{E}} \mu(\bar{D})$ as follows:

$$\nabla^+_{\bar{E}} \mu(\bar{D}) = \lim_{\varepsilon \to 0+} \frac{\mu(\bar{D} + \varepsilon\bar{E}) - \mu(\bar{D})}{\varepsilon} = (d+1)\widehat{\deg}(\bar{D}^d \cdot \bar{E}).$$

The function extends naturally to the whole space $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ of adelic arithmetic $\mathbb{R}$-Cartier divisors (see Section 1.1.7) and thus defines a map $\nabla_\mu : \widehat{\mathrm{Div}}_{\mathbb{R}}(X) \times C_\circ \to \mathbb{R}$. Note that the subset $C_{\circ\circ}$ of $\bar{D} \in C_\circ$ such that $\nabla_\mu(\cdot, \bar{D})$ preserves the order contains all nef adelic arithmetic $\mathbb{R}$-Cartier divisors. For $\bar{D} \in C_{\circ\circ}$ and any place in $v \in M_K \cup K(\mathbb{C})$, the map $\nabla_\mu(\cdot, \bar{D})$ defines a positive functional $\Psi^{\widehat{\deg}}_{\bar{D},v}$ on $C^0(X_v^{\mathrm{an}})_+$ which sends $f_v \in C^0(X_v^{\mathrm{an}})$ to $\widehat{\deg}(\bar{D}^d \cdot \bar{O}(f_v))$. It is an additive functional on $C^0(X_v^{\mathrm{an}})_+$, which coincides with the functional $(\bar{D}^d)_v$ defined in Section 1.1.7 when $\bar{D}$ is nef. Therefore, from Theorem 6.2, we obtain the following corollary.

Corollary 6.12. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X. Assume that $\bar{D}$ is nef and $\widehat{\deg}(\bar{D}^{d+1}) = 0$. If s is an element of $\mathrm{Rat}(X)^\times_{\mathbb{R}}$ with $\bar{D}' := \bar{D} + \widehat{(s)} \geq 0$, then, for any $v \in M_K \cup K(\mathbb{C})$,

$$\mathrm{Supp}(\Psi^{\widehat{\deg}}_{\bar{D},v}) \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} = \emptyset. \qquad \square$$

7 Comparison of the Functionals

Let $\pi : X \to \mathrm{Spec}\,K$ be a projective and geometrically integral scheme defined over a number field K and $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X such that D is big. In view of the applications of Theorem 6.2 to different functionals, notably Corollaries 6.7, 6.11, and 6.12, a natural question is the comparison between the functionals $\Psi^{\hat{\mu}^{\mathrm{asy}}_{\max}}_{\bar{D},v}$, $\Psi^{\widehat{\mathrm{vol}}}_{\bar{D},v}$, and $\Psi^{\widehat{\deg}}_{\bar{D},v}$. For this purpose, we relate the function $\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D})$ to the graded linear series of $\bar{D}$ and show that it is always bounded from below by $\widehat{\mathrm{vol}}(\bar{D})/(d+1)\mathrm{vol}(D)$.

7.1 Asymptotic maximal slope and graded linear series

If $\bar{D} = (D, g)$ is an adelic arithmetic $\mathbb{R}$-Cartier divisor on X, we denote by $V(\bar{D})$ the K-vector subspace of $\mathrm{Rat}(X)$ generated by $\hat{H}^0(X, \bar{D})$ (see Section 1.1.8). Let ζ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}\,K$ such that $\widehat{\deg}(\zeta) = 1$. For any integer $n \geqslant 1$ and any real number t, we denote by $V_n^{\zeta,t}(\bar{D})$ the K-vector subspace of $R(X)$ defined as

$$V_n^{\zeta,t}(\bar{D}) := V(n\bar{D} - \pi^*(nt\zeta)).$$

The direct sum $V_\bullet^{\zeta,t}(\bar{D})$ forms a graded sub-K-algebra of $V_\bullet(D) := \bigoplus_{n\geq 0} H^0(X, nD)$. Moreover, $(V_\bullet^{\zeta,t})_{t\in\mathbb{R}}$ forms a multiplicatively concave family of graded linear series of the $\mathbb{R}$-divisor D. Namely, for $(t_1, t_2) \in \mathbb{R}^2$ and $(n, m) \in \mathbb{N}^2$ one has

$$V_n^{\zeta,t_1}(\bar{D}) V_m^{\zeta,t_2}(\bar{D}) \subset V_{n+m}^{\zeta,t}(\bar{D}), \tag{7.1}$$

where $t = (nt_1 + mt_2)/(n + m)$.

For any integer $n \geq 1$, let $\lambda_n^\zeta(\bar{D})$ be the supremum of the set

$$\{t \in \mathbb{R} \mid V_n^{\zeta,t}(\bar{D}) \neq \{0\}\}.$$

The function $\lambda_n^\zeta(.)$ preserves the order. Namely, if $\bar{D}$ and $\bar{D}'$ are two adelic arithmetic $\mathbb{R}$-Cartier divisors on X such that $\bar{D} \leq \bar{D}'$, then one has $\lambda_n^\zeta(\bar{D}) \leq \lambda_n^\zeta(\bar{D}')$. Therefore, by [3, Lemma 2.6] (the Hermitian line bundle case), the sequence $(\lambda_n^\zeta(\bar{D}))_{n\geq 1}$ is bounded from above. Moreover, the relation (7.1) shows that the sequence $(n\lambda_n^\zeta(\bar{D}))_{n\geq 1}$ is super-additive. Therefore, the sequence $(\lambda_n^\zeta(\bar{D}))_{n\geq 1}$ converges in $\mathbb{R}$. The following proposition relate the limit of the sequence $(\lambda_n^\zeta(\bar{D}))_{n\geq 1}$ to the asymptotic maximal slope of $\bar{D}$. In particular, the function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$ coincides with the one defined in [7, Section 4.2] in the adelic line bundle setting.

Proposition 7.1. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is big, then for any $\zeta \in \widehat{\mathrm{Div}}_\mathbb{R}(\mathrm{Spec}K)$ with $\widehat{\deg}(\zeta) = 1$ one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = \lim_{n\to+\infty} \lambda_n^\zeta(\bar{D}). \qquad \square$$

Proof. We say that an adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{E}$ satisfies to the $\mathbb{Q}$-Dirichlet property if there is $\varphi \in \mathrm{Rat}(X)_\mathbb{Q}^\times$ such that $\bar{E} + \widehat{(\varphi)} \geq 0$. This condition is stronger than the usual Dirichlet property. We define $\hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D})$ as

$$\sup\{t \in \mathbb{R} \mid \bar{D} - t\pi^*(\zeta) \text{ has the } \mathbb{Q} - \text{Dirichlet property}\}.$$

(1) $\lim_{n\to+\infty} \lambda_n^\zeta(\bar{D}) \leq \hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D})$: If $n\bar{D} - tn\pi^*(\zeta) + \widehat{(\phi)} \geq 0$ for some $\phi \in \mathrm{Rat}(X)^\times$, then $\bar{D} - t\pi^*(\zeta)$ has the $\mathbb{Q}$-Dirichlet property, so that $\lambda_n^\zeta(\bar{D}) \leq \hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D})$, and hence the inequality follows.

(2) $\lim_{n\to+\infty} \lambda_n^\zeta(\bar{D}) \geq \hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D})$: Let ϵ be a positive number. Then there is $t \in \mathbb{R}$ such that

$$\hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D}) - \epsilon \leq t \leq \hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D})$$

and $\bar{D} - t\pi^*(\zeta)$ has the $\mathbb{Q}$-Dirichlet property. Therefore, $n\bar{D} - tn\pi^*(\zeta) + (\phi) \geq 0$ for some $n \in \mathbb{Z}_{>0}$ and $\phi \in \mathrm{Rat}(X)^\times$. Thus,

$$\hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D}) - \epsilon \leq t \leq \lambda_n^\zeta(\bar{D}) \leq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}),$$

as required.

(3) $\hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D}) \leq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$: This is obvious.

(4) $\hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D}) \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$: Let ϵ be a positive number. Let ζ_ϵ be the adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}K$ given by $\zeta_\epsilon := \sum_{\sigma \in K(\mathbb{C})} \epsilon[\sigma]$. We assume that $\bar{D} - t\pi^*(\zeta)$ has the Dirichlet property. In particular, $\bar{D} - t\pi^*(\zeta)$ is pseudo-effective, so that $\bar{D} + \pi^*(\zeta_\epsilon) - t\pi^*(\zeta)$ is big by [17, Proposition 4.4.2(3)], so that $\bar{D} + \pi^*(\zeta_\epsilon) - t\pi^*(\zeta)$ has the $\mathbb{Q}$-Dirichlet property. Therefore, we have

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \leq \hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D} + \pi^*(\zeta_\epsilon)) = \hat{\mu}_{\max,\mathbb{Q}}^{\mathrm{asy}}(\bar{D}) + \epsilon[K:\mathbb{Q}]/2,$$

as desired.   ∎

The following proposition compares the arithmetic maximal slope to a normalized form of the arithmetic volume function.

Proposition 7.2. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is big and $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq 0$. Then one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq \frac{\widehat{\mathrm{vol}}(\bar{D})}{(d+1)\mathrm{vol}(D)}. \tag{7.2}$$

In particular, if $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = \widehat{\mathrm{vol}}(\bar{D}) = 0$, then for any $\bar{E} \in \widehat{\mathrm{Div}}_\mathbb{R}(X)_+$ one has

$$\nabla_{\bar{E}}^+ \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq \frac{1}{(d+1)\mathrm{vol}(D)} \nabla_{\bar{E}}^+ \widehat{\mathrm{vol}}(\bar{D}), \tag{7.3}$$

so that, for any $v \in M_K \cup K(\mathbb{C})$, one has

$$\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}(f_v) \geq \frac{1}{(d+1)\mathrm{vol}(D)} \Psi_{\bar{D},v}^{\widehat{\mathrm{vol}}}(f_v) \tag{7.4}$$

for any nonnegative continuous function f_v on X_v^{an} and hence

$$\mathrm{Supp}(\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}) \supseteq \mathrm{Supp}(\Psi_{\bar{D},v}^{\widehat{\mathrm{vol}}}). \qquad \square$$

Proof.   Let ζ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}\,K$ such that $\widehat{\deg}(\zeta) = 1$. By [3, Corollary 1.13], one has

$$\widehat{\mathrm{vol}}(\bar{D}) = (d+1) \int_0^{+\infty} \mathrm{vol}(V_\bullet^{\zeta,t})\,\mathrm{d}t.$$

Therefore,

$$\frac{\widehat{\mathrm{vol}}(\bar{D})}{(d+1)\mathrm{vol}(D)} = \int_0^{+\infty} \frac{\mathrm{vol}(V_\bullet^{\zeta,t})}{\mathrm{vol}(V_\bullet(D))}\,\mathrm{d}t.$$

Moreover, by Proposition 7.1 we obtain that $V_n^{\zeta,t} = \{0\}$ once $n \geq 1$ and $t > \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$. Therefore, (7.2) is proved. The equality (7.3) comes from (7.2) and the definition of $\nabla_{\bar{E}}^+$, and (7.4) is a special case of (7.3). ∎

7.2 Poicaré–Lelong formula and integration of Green function

Let X be a geometrically integral projective scheme of dimension d defined over a number field K, and $\bar{D}_1, \ldots, \bar{D}_d$ be integrable adelic arithmetic $\mathbb{R}$-Cartier divisors on X, and $\bar{D} = (D, g)$ be an arbitrary adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. The purpose of this section is to establish the following result.

Proposition 7.3.   For each place $v \in M_K \cup K(\mathbb{C})$, the Green function g_v is integrable with respect to the signed measure $\bar{D}_1 \cdots \bar{D}_d$. Moreover, if $[D]$ denotes the $\mathbb{R}$-coefficient algebraic cycle of dimension $d-1$ corresponding to D, then the following relation holds (see Section 1.1.7)

$$h(\bar{D}_1, \ldots, \bar{D}_d, \bar{D}; X) = h(\bar{D}_1, \ldots, \bar{D}_d; [D]) + \sum_{v \in M_K \cup K(\mathbb{C})} (\bar{D}_1 \cdots \bar{D}_d)_v(g_v). \tag{7.5}$$
□

Note that in the particular case where $D_1, \ldots, D_d, D$ are Cartier divisors, this result has been obtained in [14, Section 5] and [6], respectively, for Hermitian and adelic cases.

Before proving the above proposition, we present several observations as follows. Let $\bar{D}_1, \ldots, \bar{D}_d$ be integrable adelic arithmetic $\mathbb{R}$-Cartier divisors on X, and $\mathfrak{p}$ be a place in M_K. In [17], the number $\widehat{\deg}_\mathfrak{p}(\bar{D}_1 \cdots \bar{D}_d; \phi)$ was defined for an integrable continuous function ϕ on $X_\mathfrak{p}^{\mathrm{an}}$. Moreover, the set of integrable continuous functions is dense in $C^0(X_\mathfrak{p}^{\mathrm{an}})$ with respect to the supremum norm. Therefore, one has a natural extension of the functional $\log \#(O_K/\mathfrak{p})\widehat{\deg}_\mathfrak{p}(\bar{D}_1 \cdots \bar{D}_d; \phi)$ for any continuous function ϕ, which defines a signed Borel measure on $X_\mathfrak{p}^{\mathrm{an}}$ which we denote by $(\bar{D}_1 \cdots \bar{D}_d)_\mathfrak{p}$. Similarly, for any $\sigma \in K(\mathbb{C})$, the product of currents $\frac{1}{2}c_1(D_1, g_{1,\sigma}) \wedge \cdots \wedge c_1(D_d, g_{d,\sigma})$ defines a signed Borel

measure on X_σ^{an} which we denote by $(\bar{D}_1 \cdots \bar{D}_d)_\sigma$. See Section 1.1.7 for a presentation of the construction of these measures in the language of arithmetic intersection product. In particular, the proposition is true in the special case where $D = 0$.

Proof of Proposition 7.3. Note that both sides of the equality (7.5) are multilinear with respect to the vector $(\bar{D}_1, \ldots, \bar{D}_d, \bar{D})$. Therefore, we may assume that $\bar{D}_1, \ldots, \bar{D}_d$ are relatively nef and D is ample without loss of generality. Moreover, for any place $v \in M_K \cup K(\mathbb{C})$ two D-Green functions on X_v^{an} differ by a continuous function on X_v^{an}. Therefore, it suffices to prove the proposition for an arbitrary choice of adelic D-Green functions, and the general case follows by the linearity of the problem and the particular case where $D = 0$. In particular, we can assume without loss of generality that $\bar{D}$ is a *relatively ample arithmetic Cartier divisor*, namely D comes from an ample line bundle L on X and the adelic structure on D comes from an ample integral model of (X, L) equipped with semi-positive metrics at infinite places.

We shall prove the following claim by induction on k. Note that the case where $k = d + 1$ is just the result of the proposition itself.

Claim 7.4. Assume that $\bar{D}_i$ is relatively nef for any $i \in \{1, \ldots, d\}$. Let k be an index in $\{1, \ldots, d + 1\}$. Then the assertion of the proposition holds provided that each $\bar{D}_i$ $(k \le i \le d)$ can be written as a positive linear combination of ample Cartier divisors equipped with Green functions of $C^0 \cap \mathrm{PSH}$-type. $\qquad\square$

The claim in the case where $k = 1$ is classical, which results from [6, Theorem 4.1] by multilinearity. In the following, we verify that the claim for k implies the same claim for $k + 1$. We choose an $\mathbb{R}$-Cartier divisor E_k such that E_k can be written as a positive linear combination of ample Cartier divisors and $D'_k = E_k + D_k$ is an ample Cartier divisor. We choose suitable E_k-Green functions such that $\bar{E}_k$ can be written as a positive linear combination of ample Cartier divisors equipped with Green functions of $C^0 \cap \mathrm{PSH}$-type. Then $\bar{D}'_k = \bar{E}_k + \bar{D}_k$ is an ample Cartier divisor equipped with Green functions of $C^0 \cap \mathrm{PSH}$-type. The induction hypothesis then implies that the claim holds for $\bar{D}_1, \ldots, \bar{D}_{k-1}, \bar{E}_k, \bar{D}_{k+1}, \ldots, \bar{D}_d$ and for $\bar{D}_1, \ldots, \bar{D}_{k-1}, \bar{D}'_k, \bar{D}_{k+1}, \ldots, \bar{D}_d$. We then conclude by the multilinearity of the problem. $\qquad\blacksquare$

7.3 Intersection measure and comparison with $\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}$

Similarly to the results obtained in the previous subsection, in the case where $\bar{D}$ is nef and D is big, the linear functional $((d+1)\mathrm{vol}(D))^{-1}\Psi_{\bar{D},v}^{\widehat{\deg}}$ is bounded from above by $\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}$

provided that $\widehat{\deg}(\bar{D}^{d+1}) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = 0$ (we can actually prove that they are equal). This comparison uses a generalization of the positive intersection product to the framework of adelic arithmetic $\mathbb{R}$-Cartier divisors.

Let $\bar{D}$ be a big adelic arithmetic $\mathbb{R}$-Cartier divisor on X. We denote by $\Theta(\bar{D})$ the set of couples $(\nu, \bar{N})$, where $\nu : X' \to X$ is a birational projective morphism and $\bar{N}$ is a nef adelic arithmetic $\mathbb{R}$-Cartier divisor on X' such that

$$\hat{H}^0(X', t(\nu^*(\bar{D}) - \bar{N})) \neq \{0\}$$

for some $t > 0$. We then define a functional $\langle \bar{D}^d \rangle$ on the cone $\widehat{\mathrm{Nef}}_{\mathbb{R}}(X)$ of nef adelic arithmetic $\mathbb{R}$-Cartier divisors on X as

$$\forall \bar{A} \in \widehat{\mathrm{Nef}}_{\mathbb{R}}(X), \quad \langle \bar{D}^d \rangle \cdot \bar{A} := \sup_{(\nu, \bar{N}) \in \Theta} \widehat{\deg}(\bar{N}^d \cdot \nu^*(\bar{A})).$$

If the set $\Theta(\bar{D})$ is empty, then the value of $\langle \bar{D}^d \rangle \cdot \bar{A}$ is defined to be zero by convention. The set $\Theta(\bar{D})$ is preordered in the following way:

$$(\nu_1 : X_1 \to X, \bar{N}_1) \geq (\nu_2 : X_2 \to X, \bar{N}_2)$$

if and only if there exists a birational modification $\nu' : X' \to X$ over both X_1 and X_2 such that $\hat{H}^0(X', t(p_1^* \bar{N}_1 - p_2^* \bar{N}_2)) \neq \{0\}$ for some $t > 0$, where $p_i : X' \to X_i$ ($i = 1, 2$) are structural morphisms (which are birational projective morphisms such that $\nu_1 p_1 = \nu_2 p_2 = \nu'$). By the same method of [9, Proposition 3.3; 12, Section 3], one can prove that $\Theta(\bar{D})$ is filtered with respect to this preorder and hence $\langle \bar{D}^d \rangle$ is an additive and positively homogeneous functional on $\widehat{\mathrm{Nef}}_{\mathbb{R}}(X)$ and hence extends to a linear form on the vector space $\widehat{\mathrm{Int}}_{\mathbb{R}}(X)$ of integrable adelic arithmetic $\mathbb{R}$-Cartier divisors. Finally, by [17, Theorem 3.3.7], one can extend by continuity the functional $\langle \bar{D}^d \rangle$ to the whole space $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ of adelic arithmetic $\mathbb{R}$-Cartier divisors such that $\langle \bar{D}^d \rangle \cdot \bar{E} \geq 0$ if $\bar{E} \geq 0$.

The comparison between $\mathrm{vol}(D)^{-1} \Psi_{\bar{D}, v}^{\widehat{\deg}}$ and $\Psi_{\bar{D}, v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}$ comes from the following variant of the arithmetic analog of Siu's inequality [21, Theorem 2.2] in the adelic arithmetic $\mathbb{R}$-Cartier divisor setting.

Theorem 7.5. Let $\bar{D} = (D, g)$ and $\bar{L} = (L, h)$ be nef adelic arithmetic $\mathbb{R}$-Cartier divisors. Then

$$\widehat{\mathrm{vol}}(\bar{D} - \bar{L}) \geq \widehat{\deg}(\bar{D}^{d+1}) - (d+1)\widehat{\deg}(\bar{D}^d \cdot \bar{L}). \tag{7.6}$$

$\square$

Proof.   By [17, Definition 2.1.6 and Proposition 2.1.7], there are sequences

$$\{(\mathscr{X}_n, \mathscr{D}_n)\}_{n=1}^{\infty} \quad \text{and} \quad \{(\mathscr{X}_n, \mathscr{L}_n)\}_{n=1}^{\infty}$$

of models (X, D) and (X, L), respectively, with the following properties:

1. $\bar{\mathscr{D}}_n = (\mathscr{D}_n, \sum_{\sigma \in K(\mathbb{C})} g_\sigma)$ and $\bar{\mathscr{L}}_n = (\mathscr{L}_n, \sum_{\sigma \in K(\mathbb{C})} h_\sigma)$ are nef for each n.
2. If we set

$$\left(0, \sum_{\mathfrak{p} \in M_K} g'_{n,\mathfrak{p}}[\mathfrak{p}]\right) = \bar{D} - \bar{\mathscr{D}}_n^{\mathrm{a}} \quad \text{and} \quad \left(0, \sum_{\mathfrak{p} \in M_K} h'_{n,\mathfrak{p}}[\mathfrak{p}]\right) = \bar{L} - \bar{\mathscr{L}}_n^{\mathrm{a}},$$

then $\lim_{n\to\infty} \|g'_{n,\mathfrak{p}}\|_{\sup} = 0$ and $\lim_{n\to\infty} \|h'_{n,\mathfrak{p}}\|_{\sup} = 0$.

Therefore, by [17, Theorem 5.2.1], it is sufficient to see the case where $\bar{D} = \bar{\mathscr{D}}$ and $\bar{L} = \bar{\mathscr{L}}$ for nef arithmetic $\mathbb{R}$-Cartier divisors $\bar{\mathscr{D}}$ and $\bar{\mathscr{L}}$ on some arithmetic variety $\mathscr{X}$.

Let $\bar{\mathscr{A}}$ be an ample arithmetic $\mathbb{R}$-Cartier divisor on $\mathscr{X}$. If the assertion of the theorem holds for $\bar{\mathscr{D}} + \epsilon\bar{\mathscr{A}}$ and $\bar{\mathscr{L}} + \epsilon\bar{\mathscr{A}}$ ($\epsilon > 0$), then, by using the continuity of the volume function, we have the assertion for our case, so that we may assume that $\mathscr{D}$ and $\mathscr{L}$ are ample. Thus, we can set

$$\mathscr{D} = a_1 \mathscr{D}'_1 + \cdots + a_l \mathscr{D}'_l \quad \text{and} \quad \mathscr{L} = b_1 \mathscr{L}'_1 + \cdots + b_r \mathscr{L}'_r,$$

where $\mathscr{L}'_1, \ldots, \mathscr{L}'_l, \mathscr{D}'_1, \ldots, \mathscr{D}'_r$ are ample Cartier divisors on $\mathscr{X}$ and

$$a_1, \ldots, a_l, b_1, \ldots, b_r \in \mathbb{R}_{>0}.$$

Let g'_i (respectively, h'_j) be a $\mathscr{D}'_i$-Green function (respectively, $\mathscr{L}'_j$-Green function) such that $\bar{\mathscr{D}}'_i = (\mathscr{D}'_i, g'_i)$ (respectively, $\bar{\mathscr{L}}'_j = (\mathscr{L}'_j, h'_j)$) is ample. We set

$$\bar{\mathscr{D}} = a_1 \bar{\mathscr{D}}'_1 + \cdots + a_l \bar{\mathscr{D}}'_l + (0, \phi) \quad \text{and} \quad \bar{\mathscr{L}} = b_1 \bar{\mathscr{L}}'_1 + \cdots + b_r \bar{\mathscr{L}}'_r + (0, \psi).$$

Moreover, for $a'_1, \ldots, a'_l, b'_1, \ldots, b'_r \in \mathbb{R}$, we set

$$\bar{\mathscr{D}}_{a'_1, \ldots, a'_l} = a'_1 \bar{\mathscr{D}}'_1 + \cdots + a'_l \bar{\mathscr{D}}'_l + (0, \phi) \quad \text{and} \quad \bar{\mathscr{L}}_{b'_1, \ldots, b'_r} = b'_1 \bar{\mathscr{L}}'_1 + \cdots + b'_r \bar{\mathscr{L}}'_r + (0, \psi).$$

Note that if $a'_1 \geq a_1, \ldots, a'_l \geq a_l$ and $b'_1 \geq b_1, \ldots, b'_r \geq b_r$, then $\bar{\mathscr{D}}_{a'_1, \ldots, a'_l}$ and $\bar{\mathscr{L}}_{b'_1, \ldots, b'_r}$ are nef, so that, using the continuity of the volume function together with the arithmetic analog of Siu's inequality (cf. [21, Theorem 2.2]) for nef arithmetic $\mathbb{Q}$-divisors, we have the assertion. ∎

Corollary 7.6. Let $\bar{D}$ be a relatively nef adelic arithmetic $\mathbb{R}$-Cartier divisor on X such that D is big. If

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = \frac{\widehat{\deg}(\bar{D}^{d+1})}{(d+1)\mathrm{vol}(D)}, \tag{7.7}$$

then one has

$$\forall \bar{E} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X), \quad \nabla_{\bar{E}}^{+}\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = \frac{\widehat{\deg}(\bar{D}^d \cdot \bar{E})}{\mathrm{vol}(D)}. \tag{7.8}$$

In particular,

$$\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}} = \frac{(\bar{D}^d)_v}{\mathrm{vol}(D)} \tag{7.9}$$

for any $v \in M_K \cup K(\mathbb{C})$. $\qquad\square$

Proof. Since the map $\bar{E} \mapsto \nabla_{\bar{E}}^{+}\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$ from $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ to $\mathbb{R} \cup \{+\infty\}$ is super-additive and $\bar{E} \mapsto \widehat{\deg}(\bar{D}^d \cdot \bar{E})$ is a linear functional, it suffices to establish the inequality (see [9, Remark 4.3])

$$\forall \bar{E} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X), \quad \nabla_{\bar{E}}^{+}\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq \frac{\widehat{\deg}(\bar{D}^d \cdot \bar{E})}{\mathrm{vol}(D)}.$$

Note that both the condition (7.7) and the assertion (7.9) remain equivalent if one replaces $\bar{D}$ by $\bar{D} + \pi^*(\zeta)$ with $\zeta \in \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathrm{Spec}K)$. Therefore, we may assume that $\bar{D}$ is nef and big without loss of generality. In this case, one has

$$\forall \bar{E} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X), \quad \widehat{\deg}(\bar{D}^d \cdot \bar{E}) = \langle \bar{D}^d \rangle \cdot \bar{E}$$

since $\bar{D}$ is nef. We shall actually establish the equality

$$\nabla_{\bar{E}}^{+}\widehat{\mathrm{vol}}(\bar{D}) = (d+1)\langle \bar{D}^d \rangle \cdot \bar{E}$$

for any $\bar{E} \in \widehat{\mathrm{Int}}_{\mathbb{R}}(X)$. We choose $\bar{M} \in \widehat{\mathrm{Nef}}_{\mathbb{R}}(X)$ such that $\bar{M} - \bar{E}$ and $\bar{M} + \bar{E}$ are nef and big and $\bar{M} - \bar{D}$ is big. Then for any $(v, \bar{N}) \in \Theta(\bar{D})$, by (7.6) one has

$$\widehat{\mathrm{vol}}(\bar{D} + t\bar{E}) \geq \widehat{\mathrm{vol}}(\bar{N} + tv^*(\bar{E})) = \widehat{\mathrm{vol}}((\bar{N} + tv^*(\bar{M})) - tv^*(\bar{M} - \bar{E}))$$

$$\geq \widehat{\deg}((\bar{N} + tv^*(\bar{M})^{d+1}) - (d+1)t\widehat{\deg}((\bar{N} + tv^*(\bar{M}))^d \cdot v^*(\bar{M} - \bar{E}))$$

$$= \widehat{\deg}(\bar{N}^{d+1}) + t(d+1)\widehat{\deg}(\bar{N}^d \cdot \bar{E}) + O(t^2), \tag{7.10}$$

where the implicit constant in $O(t^2)$ only depends on $\widehat{\mathrm{vol}}(\bar{M}) = \widehat{\deg}(v^*(\bar{M})^{d+1})$. We then deduce

$$\widehat{\mathrm{vol}}(\bar{D} + t\bar{E}) \geq \widehat{\mathrm{vol}}(\bar{D}) + t(d+1)\langle \bar{D}^d \rangle \cdot \bar{E} + O(t^2),$$

which implies that

$$\nabla^{+}_{\bar{E}}\widehat{\mathrm{vol}}(\bar{D}) \geq (d+1)\langle \bar{D}^d \rangle \cdot \bar{E}.$$

By the continuity of the linear functional $\bar{E} \mapsto \langle \bar{D}^d \rangle \cdot \bar{E}$, we obtain that this inequality holds for general $\bar{E} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$. Finally, by the log-concavity of the arithmetic volume function, the functional $\bar{E} \mapsto \nabla^{+}_{\bar{E}}\widehat{\mathrm{vol}}(\bar{D})$ is super-additive (see [9, Remark 4.2]). Therefore, one obtains $\nabla^{+}_{\bar{E}}\widehat{\mathrm{vol}}(\bar{D}) = (d+1)\langle \bar{D}^d \rangle \cdot \bar{E}$. The result is thus proved by using the relation (7.3).   ∎

7.4   Comparison with the distribution of nonpositive points

In this subsection, we compare the functional approach and the distribution of nonpositive points in the particular case where the set of nonpositive points is Zariski dense. The main point is an equidistribution argument. Let X be a projective and geometrically integral variety over a number field K. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X. We assume that $\bar{D}$ is nef and D is big.

Proposition 7.7. Assume that the set $X(\bar{K})^{\bar{D}}_{\leq 0}$ is Zariski dense, then for any place $v \in M_K \cup K(\mathbb{C})$ one has

$$\Psi^{\hat{\mu}^{\mathrm{asy}}_{\max}}_{\bar{D},v} = \frac{(\bar{D}^d)_v}{\mathrm{vol}(D)}$$

and

$$\mathrm{Supp}_{\mathrm{ess}}(X(\bar{K})^{\bar{D}}_{\leq 0})^{\mathrm{an}}_v \supseteq \mathrm{Supp}\,\Psi^{\hat{\mu}^{\mathrm{asy}}_{\max}}_{\bar{D},v}. \qquad \square$$

Proof.   Since the set $X(\bar{K})^{\bar{D}}_{\leq 0}$ is Zariski dense, we obtain that the essential minimum of the height function $h_{\bar{D}}(.)$ is nonpositive. However, since $\bar{D}$ is nef one has $\widehat{\deg}(\bar{D}^{d+1}) = \widehat{\mathrm{vol}}(\bar{D})$ and therefore

$$0 = \hat{\mu}_{\mathrm{ess}}(\bar{D}) \geq \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}) \geq \frac{\widehat{\mathrm{vol}}(\bar{D})}{(d+1)\mathrm{vol}(D)} \geq 0,$$

which implies that $\widehat{\deg}(\bar{D}^{d+1}) = \widehat{\mathrm{vol}}(\bar{D}) = 0$.

Let $S = (x_n)_{n\geq 1}$ be a generic sequence in $X(\bar{K})^{\bar{D}}_{\leq 0}$. For any adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{E}$ on X, we define

$$\Phi_S(\bar{E}) := \liminf_{n \to +\infty} h_{\bar{E}}(x_n).$$

This function takes value in $\mathbb{R} \cup \{+\infty\}$ on the cone Θ of adelic arithmetic $\mathbb{R}$-Cartier divisors $\bar{E}$ such that E is big. The function $\Phi_S(.) : \Theta \to \mathbb{R} \cup \{+\infty\}$ is also super-additive. Moreover, one has $\Phi_S(\bar{E}) \geq \hat{\mu}_{\max}^{\text{asy}}(\bar{E})$ for any $\bar{E} \in \Theta$ and $\Phi_S(\bar{D}) = \hat{\mu}_{\max}^{\text{asy}}(\bar{D}) = 0$. Therefore, one has

$$\forall \bar{E} \in \widehat{\text{Div}}_{\mathbb{R}}(X), \quad \nabla_{\bar{E}}^{+} \Phi_S(\bar{D}) \geq \nabla_{\bar{E}}^{+} \hat{\mu}_{\max}^{\text{asy}}(\bar{D}).$$

By (7.8) and [9, Proposition 4.3], one obtains

$$\forall \bar{E} \in \widehat{\text{Div}}_{\mathbb{R}}(X), \quad \nabla_{\bar{E}}^{+} \Phi_S(\bar{D}) = \nabla_{\bar{E}}^{+} \hat{\mu}_{\max}^{\text{asy}}(\bar{D}) = \frac{\widehat{\deg}(\bar{D}^d \cdot \bar{E})}{\text{vol}(D)}.$$

This relation implies that, for any $\bar{E} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, the sequence $(h_{\bar{E}}(x_n))_{n \geq 1}$ actually converges to $\text{vol}(D)^{-1} \widehat{\deg}(\bar{D}^d \cdot \bar{E})$. In fact, one has $h_{\bar{D}}(x_n) = 0$ for any $n \in \mathbb{N}$, $n \geq 1$. Therefore, $\nabla_{\bar{E}}^{+} \Phi_S(\bar{D}) = \Phi_S(\bar{E})$. In particular, one has $\Phi_S(-\bar{E}) = -\Phi_S(\bar{E})$, which implies the convergence of the sequence $(h_{\bar{E}}(x_n))_{n \geq 1}$.

Suppose $\text{Supp}_{\text{ess}}(X(\bar{K})_{\leq 0}^{\bar{D}})_v^{\text{an}} \not\supseteq \text{Supp} \Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\text{asy}}}$, that is, there is $w_v \in \text{Supp} \Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\text{asy}}} \setminus \text{Supp}_{\text{ess}}(X(\bar{K})_{\leq 0}^{\bar{D}})_v^{\text{an}}$. As $w_v \notin \text{Supp}_{\text{ess}}(X(\bar{K})_{\leq 0}^{\bar{D}})_v^{\text{an}}$, there is a closed subscheme Y of X such that $Y \subsetneq X$ and $w_v \notin \overline{\Delta(X(\bar{K})_{\leq 0}^{\bar{D}}; Y)_v^{\text{an}}}$. Let f_v be a nonnegative continuous function on X_v^{an} such that $f_v(w_v) = 1$ and $f_v \equiv 0$ on $\overline{\Delta(X(\bar{K})_{\leq 0}^{\bar{D}}; Y)_v^{\text{an}}}$.

Claim 7.8. For $x \in X(\bar{K})_{\leq 0}^{\bar{D}} \setminus Y(\bar{K})$, we have $h_{\bar{O}(f_v)}(x) = 0$. $\qquad\square$

Proof. If $v \in M_K$, the assertion is obvious, so that assume $v \in K(\mathbb{C})$. By the definition of $\bar{O}(f_v)$ (cf. Section 1.1.5),

$$4[K(x) : K] h_{\bar{O}(f_v)}(x) = \sum_{w \in O_v(x)} f_v(w) + \sum_{w' \in O_{\bar{v}}(x)} f_v(F_\infty(w')).$$

Note that $F_\infty(O_{\bar{v}}(x)) = O_v(x)$, and hence the assertion follows. $\qquad\blacksquare$

By the previous observation,

$$\lim_{n \to \infty} h_{\bar{O}(f_v)}(x_n) = \Phi_S(\bar{O}(f_v)) = \frac{(\bar{D}^d)_v(f_v)}{\text{vol}(D)} > 0.$$

On the other hand, as $S = (x_n)_{n \geq 1}$ is generic, there is a subsequence $S' = (x_{n_i})$ such that $x_{n_i} \notin Y(\bar{K})$ for all i, so that, by the above claim, $\lim_{i \to \infty} h_{\bar{O}(f_v)}(x_{n_i}) = 0$. This is a contradiction. $\qquad\blacksquare$

8 Extension of the Asymptotic Maximal Slope

In this section, we extend the function of the asymptotic maximal slope to the whole space $\widehat{\text{Div}}_{\mathbb{R}}(X)$ of adelic arithmetic $\mathbb{R}$-Cartier divisors. Let $\bar{D}$ be an adelic arithmetic

$\mathbb{R}$-Cartier divisor on X. We define $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$ to be

$$\inf_{\bar{D}_0 \in \Theta} \lim_{t \to +\infty} (\hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_0)) \in \mathbb{R} \cup \{-\infty\}, \qquad (8.1)$$

where Θ denotes the set of all adelic arithmetic $\mathbb{R}$-Cartier divisors $\bar{E}$ such that E is big. Note that if D is big, then the value (8.1) coincides with the maximal asymptotic maximal slope of $\bar{D}$. In fact, for any $\bar{D}_0 \in \Theta$ one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_0) \geq \hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}_0) + \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_0) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}).$$

Therefore, the infimum is attained at $\bar{D}_0 = \bar{D}$ and coincides with $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$.

The extended function also verifies the good properties such as positive homogenity, super-additivity etc. We resume these properties in the following proposition.

Proposition 8.1.

(1) Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor on X. For any $\lambda \geq 0$, one has $\hat{\mu}_{\max}^{\mathrm{asy}}(\lambda \bar{D}) = \lambda \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D})$.

(2) Let $\bar{D}_1$ and $\bar{D}_2$ be two adelic arithmetic $\mathbb{R}$-Cartier divisors on X. One has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1 + \bar{D}_2) \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1) + \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_2).$$

(3) Let $\bar{D}_1$ and $\bar{D}_2$ be two adelic arithmetic $\mathbb{R}$-Cartier divisors on X. If $\bar{D}_1 \geq \bar{D}_2$, then $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1) \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_2)$.

(4) If $\bar{D}$ is an adelic arithmetic $\mathbb{R}$-Cartier divisor on X and ζ is an adelic arithmetic $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}K$, one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + \pi^*(\zeta)) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) + \widehat{\deg}(\zeta).$$

(5) For any adelic arithmetic $\mathbb{R}$-Cartier divisor $\bar{D}$ on X and any $\varphi \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$, one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + \widehat{(\varphi)}) = \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}). \qquad \square$$

Proof.   (1) The equality is trivial when $\lambda = 0$. In the following, we assume that $\lambda > 0$. For any $\bar{D}_0 \in \Theta$, one has

$$\lim_{t \to +\infty} (\hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \lambda \bar{D}) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0))$$

$$= \lim_{t \to +\infty} (\hat{\mu}^{\mathrm{asy}}_{\max}(\lambda t \bar{D}_0 + \lambda \bar{D}) - \lambda t \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0))$$

$$= \lambda \lim_{t \to +\infty} (\hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0)).$$

By taking the infimum with respect to $\bar{D}_0$, one obtains the result.

(2) Let $\bar{D}_0$ be an element in Θ. For sufficiently positive t, one has

$$\lim_{t \to +\infty} \hat{\mu}^{\mathrm{asy}}_{\max}(2t\bar{D}_0 + \bar{D}_1 + \bar{D}_2) - 2t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0)$$

$$\geq \lim_{t \to +\infty} \hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}_1) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0) + \lim_{t \to +\infty} \hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}_1) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0)$$

$$\geq \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_1) + \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_2).$$

Since $\bar{D}_0$ is arbitrary, one obtains the result.

(3) Let $\bar{D}_0$ be an element in Θ. For sufficiently positive number t, one has

$$\lim_{t \to +\infty} \hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}_1) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0) \geq \lim_{t \to +\infty} \hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}_2) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0) \geq \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_2).$$

Since $\bar{D}_0$ is arbitrary, one obtains $\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_1) \geq \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_2)$.

(4) For any $\bar{D}_0 \in \Theta$ and any sufficiently positive number t, one has

$$\hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D} + \pi^*(\zeta)) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0) = \hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0) + \widehat{\deg}(\zeta).$$

Passing to limit when t tends to the infinity and then by taking the infimum with respect to $\bar{D}_0$, one obtains $\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D} + \pi^*(\zeta)) = \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}) + \widehat{\deg}(\zeta)$.

(5) Let $\bar{D}_0$ be an element in Θ. For sufficiently positive number t, one has

$$\hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D} + \widehat{(\varphi)}) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0) = \hat{\mu}^{\mathrm{asy}}_{\max}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}_0).$$

Therefore, $\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D} + \widehat{(\varphi)}) = \hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D})$.   ∎

The following is a criterion for the pseudo-effectivity of adelic arithmetic $\mathbb{R}$-Cartier divisors, which is a generalization of Proposition 6.9.

Proposition 8.2.   Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor. Then $\bar{D}$ is pseudo-effective if and only if D is pseudo-effective and $\hat{\mu}^{\mathrm{asy}}_{\max}(\bar{D}) \geq 0$.   □

Proof. Assume that $\bar{D}$ is pseudo-effective, then D is a pseudo-effective $\mathbb{R}$-divisor. Moreover, for any $\bar{D}_0 \in \Theta$, there exists $\zeta \in \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathrm{Spec} K)$ such that $\bar{D}_1 = \bar{D}_0 + \pi^*(\zeta)$ is big. Therefore, for any $\varepsilon > 0$, the adelic arithmetic $\mathbb{R}$-Cartier divisor $\varepsilon(\bar{D}_0 + \pi^*(\zeta)) + \bar{D}$ is big. Hence, for $t > \varepsilon$ one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_0) = \hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}_1 + \bar{D}) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1) \geq (t - \varepsilon)\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1)$$

$$= -\varepsilon\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_1).$$

Since ε is arbitrary, we obtain that

$$\lim_{t \to +\infty}(\hat{\mu}_{\max}^{\mathrm{asy}}(t\bar{D}_0 + \bar{D}) - t\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}_0)) \geq 0.$$

Conversely, assume that D is pseudo-effective and $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) \geq 0$. If $\bar{D}'$ is a big adelic arithmetic $\mathbb{R}$-Cartier divisor, then $D + D'$ is big since D is pseudo-effective and D' is big. Moreover, one has

$$\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D} + \bar{D}') \geq \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) + \hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}') > 0.$$

Hence, $\bar{D} + \bar{D}'$ is big by Proposition 6.9. Therefore, $\bar{D}$ is pseudo-effective. ∎

The results which we have obtained in Section 6.1 can be applied to the extended function $\hat{\mu}_{\max}^{\mathrm{asy}}(.)$. Let $C_\circ$ be the cone of all pseudo-effective adelic arithmetic $\mathbb{R}$-Cartier divisors and $V = \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$. Then the cone $C_\circ$ satisfies the conditions (a)–(c) of Section 6.1. Moreover, Proposition 8.2 shows that the restriction of $\hat{\mu}_{\max}^{\mathrm{asy}}$ on $C_\circ$ is a real-valued function. By Proposition 8.1, this function verifies the conditions (1)–(3) of Section 6.1. Thus, we obtain the following corollary of Theorem 6.2.

Corollary 8.3. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is pseudo-effective and that $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = 0$. If s is an element of $\mathrm{Rat}(X)_{\mathbb{R}}^\times$ with $\bar{D} + \widehat{(s)} \geq 0$, then

$$\mathrm{Supp}(\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}}) \cap \{x \in X_v^{\mathrm{an}} \mid |s|_{g_v}(x) < 1\} = \varnothing$$

for any $v \in M_K \cup K(\mathbb{C})$. □

We conclude the article by the following question.

Question 8.4. Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$-Cartier divisor such that D is big and that $\hat{\mu}_{\max}^{\mathrm{asy}}(\bar{D}) = 0$. Assume that, for any place $v \in M_K \cup K(\mathbb{C})$, the union of all algebraic curves lying in $\mathrm{Supp}(\Psi_{\bar{D},v}^{\hat{\mu}_{\max}^{\mathrm{asy}}})$ is contained in the augmented base locus of D_v, does the Dirichlet property always hold for $\bar{D}$? □

References

[1] Berkovich, V. G. *Spectral Theory and Analytic Geometry Over Non-Archimedean Fields*. Mathematical Surveys and Monographs 33. Providence, RI: America Mathematical Society, 1990.

[2] Birkar, C., P. Cascini, C. Hacon, and J. McKernan. "Existence of minimal models for varieties of log general type." *Journal of the American Mathematical Society* 23, no. 2 (2010): 405–68.

[3] Boucksom, S. and H. Chen. "Okounkov bodies of filtered linear series." *Compositio Mathematica* 147, no. 4 (2011): 1205–29.

[4] Burgos i Gil, J. I., A. Moriwaki, P. Philippon, and M. Sombra. "Arithmetic positivity on toric varieties." *Journal of Algebraic Geometry*, to appear (see also arXiv:1210.7692v1 [math.AG]).

[5] Chambert-Loir, A. "Mesures et équidistribution sur des espaces de Berkovich." *Journal für die Reine und Angewandte Mathematik* 595 (2006): 215–35.

[6] Chambert-Loir, A. and A. Thuillier. "Mesures de Mahler et équidistribution logarithmique." *Annales de l'Institut Fourier* 59, no. 3 (2009): 977–1014.

[7] Chen, H. "Convergence des polygones de Harder–Narasimhan." *Mémoire de la Société Mathématique de France* 120 (2010): 1–120.

[8] Chen, H. "Arithmetic Fujita approximation." *Annales Scientifiques de l' École Normale Supérieure* 43, no. 4 (2010): 555–78.

[9] Chen, H. "Differentiability of the arithmetic volume function." *Journal of the London Mathematical Society* 84, no. 2 (2011): 365–84.

[10] Ein, L., R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa. "Restricted volumes and base loci of linear series." *American Journal of Mathematics* 131, no. 3 (2009): 607–51.

[11] Gubler, W. "The Bogomolov conjecture for totally degenerate abelian varieties." *Inventiones Mathematicae* 169, no. 2 (2007): 377–400.

[12] Ikoma, H. "On the concavity of arithmetic volumes." *International Mathematics Research Notices* (2013): preprint arXiv:1310.8424 [math.AG].

[13] Klimek, M. *Pluripotential Theory*. London Mathematical Society Monographs, New Series 6. New York: Oxford Science Publications, 1991.

[14] Maillot, V. *Géométrie d'Arkaelov des variététs toriques et fibrés en droites intégrables*. Mémoire de la Société Mathématique de France 80. Paris: Soc. Math. France, 2000, vi+129pp.

[15]   Moriwaki, A. "Zariski decompositions on arithmetic surfaces." *Research Institute for Mathematical Sciences. Publications* 48, no. 4 (2012): 799–898.

[16]   Moriwaki, A. "Toward Dirichlet's unit theorem on arithmetic varieties." *Kyoto Journal of Mathematics* 53, no. 1 (2013): 197–259.

[17]   Moriwaki, A. "Adelic divisors on arithmetic varieties." *Memoirs of the American Mathematical Society*, to appear (see also arXiv:1302.1922 [math.AG]).

[18]   Moriwaki, A. "Toward a geometric analogue of Dirichlet's unit theorem." *Kyoto Journal of Mathematics*, to appear (see also arXiv:1311.6307 [math.AG]).

[19]   Morosawa, S., Y. Nishimura, M. Taniguchi, and T. Ueda. *Holomorphic Dynamics*. Cambridge Studies in Advanced Mathematics 66. Cambridge: Cambridge University Press, 2000.

[20]   Ullmo, E. "Positivité et discrétion des points algébriques des courbes." *Annals of Mathematics. Second Series* 147, no. 1 (1998): 167–79.

[21]   Yuan, X. "Big line bundles over arithmetic varieties." *Inventiones Mathematicae* 173, no. 3 (2007): 603–49.

[22]   Zhang, S. "Small points and adelic metrics." *Journal of Algebraic Geometry* 4, no. 2 (1995): 281–300.

[23]   Zhang, S. "Equidistribution of small points on abelian varieties." *Annals of Mathematics. Second Series* 147, no. 1 (1998): 159–65.