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# COMPUTING THE VOLUME FUNCTION ON A PROJECTIVE BUNDLE OVER A CURVE

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***Abstract.*** — We establish an explicit link between the volume function on a projective variety over a projective curve and the canonical filtrations of the direct image bundles. As an application, we compute the volume function on a projective bundle over a curve.

## Contents



## 1. Introduction

Let $k$ be a field and $X$ be a projective variety of dimension $d$ over $\operatorname{Spec} k$. For any line bundle $L$ on $X$, the *volume* of $L$ is defined as

$$\mathrm{vol}(L) := \limsup_{n\to\infty} \frac{\operatorname{rk} H^0(X, L^{\otimes n})}{n^d/d!}, \tag{1}$$

where the symbol lim sup can be replaced by lim, thanks to Fujita's approximation theorem [**9**] (see also [**14**] for the positive characteristic case). One says that $L$ is *big* if $\mathrm{vol}(L) > 0$. The volume function is invariant under the numerical equivalence (see [**10**] Proposition 2.2.41), and hence induces a mapping from the Néron-Severi group $N^1(X)$ to $\mathbb{R}_+$ which we shall still denote by vol.

If $L$ is nef, then it follows from the asymptotic Riemann-Roch formula that $\mathrm{vol}(L) = c_1(L)^d$. In particular, the volume function is polynomial on the nef cone. However, on the larger cone of big divisors, the behaviour of the volume function is much more complicated, and the volume function is difficult to calculate. It is known that, when

$X$ is a surface or a toric variety, the volume function is piecewise polynomial on the pseudo-effective cone, and can be computed by the Zariski decomposition (see [**8**] and [**12**]). For general case, some authors have proposed the *positive intersection product* for big classes which generalizes the classical intersection product and identifies $\mathrm{vol}(L)$ with the positive self-intersection product of the class of $L$ (see [**1**] for the complex analytic case and [**2**] for the algebraic case). But the positive intersection product is only super-additive in each variable, which suggests that the volume function is not a polynomial everywhere on the big cone. Although the differentiability of the volume function has been proved in [**2**] and [**11**], there does exist a counter-example where the volume function is not two-times differentiable (see [**10**], Example 2.2.46 with $n = 2$). Furthermore, the function vol may take irrational values (see [**6**]), which is not the case for the self-intersection function.

In [**5, 4**], the author has obtained an explicit link between the arithmetic volume function in sense of Moriwaki and the maximal value of the limit of Harder-Narasimhan polygons. This allows to prove a conjecture of Moriwaki on the arithmetic analogue of Fujita's approximation theorem. Although the articles were written in the arithmetic framework, a similar idea also works in function field case. Let $C$ be a smooth projective curve defined over $k$. Denote by $K := k(C)$ the field of meromorphic functions on $C$. If $E$ is a non-zero vector bundle on $C$, the Harder-Narasimhan filtration of $E$ is the unique flag

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0 \tag{2}$$

of $E$ such that each sub-quotient $E_i/E_{i+1}$ is semistable and that the successive slopes verify the following inequalities

$$\mu_0 < \mu_1 < \cdots < \mu_{d-1}, \tag{3}$$

where $\mu_i = \mu(E_i/E_{i+1})$. For any non-zero vector bundle $E$ on $C$ whose Harder-Narasimhan filtration is as (2), we define a probability measure $\nu_E$ on $\mathbb{R}$ as follows

$$\nu_E := \sum_{i=0}^{d-1} \frac{\mathrm{rk}(E_{i+1}) - \mathrm{rk}(E_i)}{\mathrm{rk}(E)} \delta_{\mu_i}, \tag{4}$$

where $\delta_x$ is the Dirac measure concentrated on $\{x\}$. For any real number $\varepsilon > 0$, we define an operator $T_\varepsilon$ on the set of all Borel probability measure on $\mathbb{R}$: if $\nu$ is such a measure and if $f$ is a continuous function of compact support on $\mathbb{R}$, then

$$\int f(x) T_\varepsilon \nu(\mathrm{d}x) = \int f(\varepsilon x) \nu(\mathrm{d}x).$$

The following is the main theorem of this article:

***Theorem 1.1.*** — *Let $\pi : X \to C$ be a projective and flat morphism, and $L$ be an arbitrary line bundle on $X$ such that $L_K$ is big. Then*

1) *the sequence of Borel probability measures $(T_{\frac{1}{n}} \nu_{\pi_*(L^{\otimes n})})_{n \geqslant 1}$ converges vaguely to a Borel probability measure $\nu_L^\pi$.*

*2) the following equality holds*

$$\mathrm{vol}(L) = \dim(X)\,\mathrm{vol}(L_K)\int x_+\nu_L^{\pi}(\mathrm{d}x),$$

*where* $x_+ = \max\{x, 0\}$.

Recall that a sequence of Borel probability measures $(\nu_n)_{n\geqslant 1}$ is said to converge *vaguely* to a Borel measure $\nu$ if, for any continuous function of compact support $f$, one has

$$\int f(x)\nu(\mathrm{d}x) = \lim_{n\to\infty}\int f(x)\nu_n(\mathrm{d}x).$$

Remind that $\nu$ need not be a probability measure. However, it is the case when the supports of $\nu_n$ are uniformly bounded.

The main ideal of the proof is to compare the rank of $\pi_*(L^{\otimes n})$ to the maximal value of its Harder-Narasimhan polygon, which coincides with the integral of $x_+$ with respect to the corresponding probability measure, multiplied by its rank.

Shortly after the first version of this article had been written, R. Lazarsfeld kindly communicated to the author Wolfe's Ph.D thesis. We observe that the idea of comparison mentioned above is quite similar to a result of Wolfe. In fact, for any non-zero vector bundle $E$ on $C$, Wolfe has compared the rank of $H^0(C,E)$ to partial sums of $\mathrm{rk}\,H^0(C,\mathrm{sq}_i E)$, where $\mathrm{sq}_i E$ is the $i^{\text{th}}$ subquotient of the Harder-Narasimhan filtration of $E$. In view of some basic facts about vector bundles on $C$ resumed in Lemma 2.1, Wolfe's result is more or less equivalent to Proposition 2.2 *infra.*

The more delicate part of the theorem is the convergence of dilated probability measures associated to direct images, which follows from a general convergence result on filtered graded algebra, established in a previous work [**5**] of the author (see also [**3**]).

Wolfe has studied in his thesis [**15**] (see also [**7**] page 9), by using his comparison result, the volume function on projective bundles over a curve. Let $E$ be a non-zero vector bundle on $C$. One uses the symbol $\mathbb{P}(E)$ to denote the $C$-scheme which represents the functor

$$\begin{array}{ccc} \mathbf{Scheme}/C & \longrightarrow & \mathbf{Set} \\ (p : X \to C) & \longmapsto & \left\{\begin{array}{c}\text{locally free quotient}\\ \text{of rank 1 of } p^*E\end{array}\right\} \end{array} \tag{5}$$

Let $\pi : \mathbb{P}(E) \to C$ be the canonical morphism. Denote by $\mathcal{O}_E(1)$ the line bundle on $\mathbb{P}(E)$ the universal quotient of $\pi^*E$ corresponding to the representable functor (5). The Picard group of $\mathbb{P}(E)$ is then generated by $\mathcal{O}_E(1)$ and $\pi^*\,\mathrm{Pic}(C)$. He proved that the Néron-Severi group $N^1(\mathbb{P}(E))$ can be separated into several sectors determined by the Harder-Narasimhan filtration of $E$ such that the volume function is polynomial on each piece.

In this article, we shall apply Theorem 1.1 to determine the volume function on $\mathbb{P}(E)$. Here we should assume that the characteristic of $k$ is zero. It suffices to compute explicitly the limit measure $\nu_{\mathcal{O}_E(1)}$, which relies on the computation of the Harder-Narasimhan filtration of $S^nE$. We calculate firstly that of $E^{\otimes n}$, which is invariant by the action of the symmetric group $\mathfrak{S}_n$, thanks to the uniqueness of

Harder-Narasimhan filtration. We then pass to the quotient by the action of $\mathfrak{S}_n$ to obtain the Harder-Narasimhan filtration of $S^nE$. We shall actually establish the following result:

***Theorem 1.2.*** — *Let $r$ be the rank of $E$. Assume that the characteristic of $k$ is zero, that the Harder-Narasimhan filtration of $E$ is as in* (2) *and that the successive slopes of $E$ are as in* (3). *Let $\underline{s} = (s_1, \cdots, s_r)$ be a vector in $\mathbb{R}^r$ such that the value $\mu_i$ appears exactly $\mathrm{rk}(E_i/E_{i+1})$ times in its coordinates. Let $\Delta \subset \mathbb{R}^r$ be the simplex defined as*

$$\{(x_1, \cdots, x_r) \mid 0 \leqslant x_j \leqslant 1, \quad x_1 + \cdots + x_r = 1\}$$

*and $\eta$ be the Lebesgue measure on $\Delta$ normalized such that $\eta(\Delta) = 1$. Then one has $\nu^{\pi}_{\mathcal{O}_E(1)} = \varphi_{\underline{s}*}\eta$, where $\varphi_{\underline{s}} : \Delta \to \mathbb{R}$ sends $(x_1, \cdots, x_r)$ to $s_1x_1 + \cdots + s_rx_r$.*

As a corollary of Theorems 1.1 and 1.2, we compute the volume of an arbitrary big line bundle on $\mathbb{P}(E)$:

***Corollary 1.3.*** — *Keep the notation of Theorem 1.2. Let $m \geqslant 1$ be an integer and $M$ be a line bundle of degree $c$ on $C$, then*

$$\mathrm{vol}(\mathcal{O}_E(m) \otimes \pi^*M) = rm^r \int \big(x + c/m\big)_+ \varphi_{\underline{s}*}\eta(\mathrm{d}x). \tag{6}$$

*where $y_+ := \max\{y, 0\}$ for any $y \in \mathbb{R}$.*

Wolfe has computed in his thesis the Harder-Narasimhan filtration of the symmetric powers of $E$ by using a more direct argument. This enabled him to prove that the volume function is piecewise polynomial. Here we propose a reformulation (20) of $\nu_{S^nE}$, which leads to a simple form of the volume function (6). This permits us to recover Wolfe's result, and furthermore, to prove that on each sector, the volume function is a homogeneous polynomial of total degree $\mathrm{rk}(E)$ on $m$ and $c$. Moreover, our method permits in principle to calculate explicitly the volume function.

The article is organized as follows: in the second section, we introduce the notion of positive degree for vector bundles on $C$, and then prove that the volume function equals to the limit of normalized positive degree, which permits to establish Theorem 1.1. In the third section, we compute the Harder-Narasimhan filtrations of tensor powers and symmetric powers of a non-zero vector bundle on $C$, and then prove Theorem 1.2 and Corollary 1.3. Finally in the fourth section, we show that Corollary 1.3 permits to recover Wolfe's result on the polynomial representation of the volume function.

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## 2. Volume function as a limit of positive degrees

We remind that all results in this section are valid in any characteristic.

**2.1. Some reminders on vector bundles.** — If $E$ is a vector bundle on $C$, we denote respectively by $h^0(E)$ and $h^1(E)$ the rank of $H^0(C,E)$ and $H^1(C,E)$ over $k$. By Serre duality, one has $h^1(E) = h^0(E^\vee \otimes \omega_C)$, where $\omega_C$ is the dualizing sheaf on $C$. Recall that the Riemann-Roch theorem predicts

$$h^0(E) - h^1(E) = \deg(E) + \operatorname{rk}(E)(1-g). \tag{7}$$

This implies in particular that the degree of $\omega_C$ is $2(g-1)$. Suppose that $E$ is non-zero. The *slope* of $E$ is defined as the quotient $\deg(E)/\operatorname{rk}(E)$, denoted by $\mu(E)$, and the *maximal slope* $\mu_{\max}(E)$ is the maximal value of the slopes of all non-zero subbundles of $E$. By definition one has the inequality $\mu(E) \leqslant \mu_{\max}(E)$. We say that $E$ is *semistable* if the equality $\mu(E) = \mu_{\max}(E)$ holds. This condition is also equivalent to the equality $\mu(E) = \mu_{\min}(E)$, where $\mu_{\min}(E)$ is the minimal value of slopes of all non-zero locally free quotient of $E$, called the *minimal slope* of $E$. As $\mu(E^\vee) = -\mu(E)$ and $\mu_{\max}(E^\vee) = -\mu_{\min}(E)$, we obtain that the semistability of $E$ is equivalent to that of $E^\vee$.

Denote by $b(C)$ the positive integer

$$\{\deg(L) \mid L \in \operatorname{Pic}(C),\ L \text{ is ample}\}. \tag{8}$$

By definition, the degree of a line bundle on $C$ is always a multiple of $b(C)$.

We recall in the following some basic facts about vector bundles on $C$.

***Lemma 2.1.*** — *Let $E$ be a non-zero vector bundle on $C$.*

*1) If $\mu_{\max}(E) < 0$, then $h^0(E) = 0$.*

*2) If $\mu_{\min}(E) > 2g-2$, then $h^0(E) = \deg(E) + \operatorname{rk}(E)(1-g)$.*

*3) If $\mu_{\min}(E) \geqslant 0$, then $|h^0(E) - \deg(E)| \leqslant \operatorname{rk}(E)(g + b(C))$.*

*Proof.* — 1) If $h^0(E) \neq 0$, then there is an injective homomorphism from $\mathcal{O}_C$ to $E$, therefore $0 = \mu(\mathcal{O}_C) \leqslant \mu_{\max}(E)$.

2) Since $\mu_{\min}(E) > 2g-2$, one has $\mu_{\max}(E^\vee \otimes \omega_C) = \deg(\omega_C) - \mu_{\min}(E) < 0$. By 1), one has $h^1(E) = h^0(E^\vee \otimes \omega_C) = 0$. So the Riemann-Roch formula implies $h^0(E) = \deg(E) + \operatorname{rk}(E)(1-g)$.

3) If $g = 0$, then, by 2), $h^0(E) = \deg(E) + \operatorname{rk}(E)$, so

$$|h^0(E) - \deg(E)| = \operatorname{rk}(E) \leqslant b(C) \operatorname{rk} E.$$

Assume in the following that $g \geqslant 1$. Let $L$ be an ample invertible $\mathcal{O}_C$-module of degree $b(C)$ and let $a = \lfloor \frac{2g-2}{b(C)} \rfloor + 1$. Then the inequalities $2g-2 < \deg(L^{\otimes a}) \leqslant 2g-2+b(C)$ hold. As $\mu_{\min}(E \otimes L^{\otimes a}) = \mu_{\min}(E) + \deg(L^{\otimes a}) > 2g-2$, we obtain

$$h^0(E) \leqslant h^0(E \otimes L^{\otimes a}) = \deg(E \otimes L^{\otimes a}) + \operatorname{rk}(E)(1-g) \leqslant \deg(E) + \operatorname{rk}(E)(g-1+b(C)).$$

On the other hand, $h^0(E) = h^1(E) + \deg(E) + \operatorname{rk}(E)(1-g) \geqslant \deg(E) + \operatorname{rk}(E)(1-g)$. Therefore, $|h^0(E) - \deg(E)| \leqslant \operatorname{rk}(E)(g-1+b(C))$. □

**2.2. Positive degree and rank of global section space.** — Let $E$ be a non-zero vector bundle on $C$ and $r$ be its rank. We introduce the notion of positive degree of $E$, which is the maximal value of the Harder-Narasimhan polygon of $E$. We then show that this value approximates $h^0(E)$.

Recall that the Harder-Narasimhan polygon $\widetilde{P}_E$ is by definition the concave function defined on $[0,r]$ whose graph is the convex hull of points of the form $(\operatorname{rk} F, \deg(F))$, where $F$ runs over all subbundles of $E$. Denote by $\deg_+(E)$ the integer $\max\limits_{x\in[0,r]} \widetilde{P}_E(x)$, called the *positive degree* of $E$.

The Harder-Narasimhan polygon can be determined from the Harder-Narasimhan flag of $E$, which is the only flag

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_i \supsetneq E_{i+1} \supsetneq \cdots \supsetneq E_d = 0 \tag{9}$$

such that the sub-quotients $E_i/E_{i+1}$ are semistable and satisfies

$$\mu(E_0/E_1) < \mu(E_1/E_2) < \cdots < \mu(E_{d-1}/E_d).$$

In fact, the vertices of the polygon $\widetilde{P}_E$ are just $(\operatorname{rk}(E_i), \deg(E_i))$.

The proposition below compares $h^0(E)$ and $\deg_+(E)$.

***Proposition 2.2.*** — *The following inequality holds:*

$$|h^0(E) - \deg_+(E)| \leqslant \operatorname{rk}(E)(g + b(C)), \tag{10}$$

*where the constant $b(C)$ is defined in* (8).

*Proof.* — Let the Harder-Narasimhan flag of $E$ be as in (9). For any integer $i$, $0 \leqslant i \leqslant d-1$, let $\mu_i = \mu(E_i/E_{i+1})$. Let $j$ bę the first index in $\{0, \cdots, d-1\}$ such that $\mu_j \geqslant 0$ (if such index does not exist, let $j = d$). By definition, the positive degree $\deg_+(E)$ coincides with $\deg(E_j)$.

If $j > 0$, then $\mu_{\max}(E/E_j) = \alpha_{j-1} < 0$, so Lemma 2.1 1) predicts that $h^0(E/E_j) = 0$; otherwise $E = E_j$ and we still have $h^0(E/E_j) = 0$. Hence $h^0(E) = h^0(E_j)$. If $j = n$, then $h^0(E_j) = 0 = \deg_+(E)$; otherwise $\mu_{\min}(E_j) = \alpha_j \geqslant 0$, and by Lemma 2.1 3),

$$|h^0(E) - \deg_+(E)| = |h^0(E_j) - \deg(E_j)| \leqslant \operatorname{rk}(E_j)(g + b(C)) \leqslant \operatorname{rk}(E)(g + b(C)).$$

□

We define the *positive slope* of $E$ to be the quotient $\mu_+(E) := \deg_+(E)/\operatorname{rk}(E)$. By definition, one has $\max(\mu_{\max}(E), 0) \geqslant \mu_+(E) \geqslant 0$.

**2.3. Volume function and positive degree of direct image.** — Let $X$ be an integral projective scheme of dimension $r$ over $\operatorname{Spec} k$ and $L$ is a line bundle on $X$. Assume that $\pi : X \to C$ is a flat $k$-morphism. Denote by $K$ the field of all rational functions on $C$. Then $X_K$ is an integral projective scheme of dimension $r-1$ over $\operatorname{Spec} K$.

***Proposition 2.3.*** — *The equality*

$$\operatorname{vol}(L) = \limsup_{n\to+\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} \tag{11}$$

*holds. Furthermore, if $L$ is big, then also is $L_K$.*

*Proof.* — By Proposition 2.2, one has

$$|\deg_+(\pi_*(L^{\otimes n})) - \mathrm{rk}_k H^0(X, L^{\otimes n})| \leqslant \mathrm{rk}_K H^0(X_K, L_K^{\otimes n})(g + b(C)).$$

Since $\mathrm{rk}_K H^0(X_K, L_K^{\otimes n}) \ll n^{r-1}$, we obtain

$$\lim_{n\to\infty} \left| \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} - \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^r/r!} \right| = 0, \tag{12}$$

which implies (11). If $L$ is big, then $\limsup_{n\to\infty} \dfrac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} > 0$. On the other hand,

$$\limsup_{n\to\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n\,\mathrm{rk}(\pi_*(L^{\otimes n}))} \leqslant \limsup_{n\to\infty} \frac{\mu_{\max}(\pi_*(L^{\otimes n}))}{n} < +\infty.$$

This implies $\limsup_{n\to\infty} \dfrac{\mathrm{rk}(\pi_*(L^{\otimes n}))}{n^{r-1}/r!} > 0$. Therefore $L_K$ is big. □

The Fujita's approximation theorem implies that the volume function is in fact a limit. More precisely, one has

$$\mathrm{vol}(L) = \lim_{n\to\infty} \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^r/r!}. \tag{13}$$

Furthermore, the formula (13) applied on $L_K$ implies

$$\mathrm{vol}(L_K) = \lim_{n\to\infty} \frac{\mathrm{rk}_K H^0(X_K, L_K)}{n^{r-1}/(r-1)!} = \lim_{n\to\infty} \frac{\mathrm{rk}(\pi_*(L^{\otimes n}))}{n^{r-1}/(r-1)!}.$$

Therefore we obtain the following equalities:

***Corollary 2.4.*** — *The following equalities hold:*

$$\mathrm{vol}(L) = \lim_{n\to\infty} \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^r/r!} = \lim_{n\to\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} = r\mathrm{vol}(L_K) \lim_{n\to\infty} \frac{\mu_+(\pi_*(L^{\otimes n}))}{n}.$$

**2.4. Proof of Theorem 1.1.** — The assertion 1) is a direct consequence of [**5**] Theorem 4.2. In the following, we shall prove the second assertion.

From Corollary 2.4, we obtain

$$\mathrm{vol}(L) = \dim(X)\,\mathrm{vol}(L_K) \lim_{n\to\infty} \frac{\mu_+(\pi_*(L^{\otimes n}))}{n},$$

where we remind that $\mu_+(\pi_*(L^{\otimes n}))$ is the maximal value of the (normalized) Harder-Narasimhan polygon of $\pi_*(L^{\otimes n})$. The assertion 2) then follows from the following lemma.

***Lemma 2.5.*** — *Let $E$ be a non-zero vector bundle of rank $r$ on $C$. The following equality holds:*

$$\mu_+(E) = \int x_+ \nu_E(\mathrm{d}x), \tag{14}$$

*where the probability measure $\nu_E$ was defined in (4).*

*Proof.* — Let $r$ be the rank of $E$,

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0$$

be the Harder-Narasimhan flag of $E$ and $\mu_0 < \cdots < \mu_{d-1}$ be its successive slopes. For any $i \in \{0, \cdots, d-1\}$, let $r_i$ be the rank of $E_i/E_{i+1}$. Assume that $j$ is the first index in $\{0, \cdots, d-1\}$ such that $\mu_j \geqslant 0$ (if such index does not exist, let $j = d$). Then

$$\deg_+(E) = \deg(E_j) = \sum_{i=0}^{j} r_i \mu_i.$$

Therefore, $\mu_+(E) = \sum_{i=0}^{j} \frac{r_i}{r}\mu_i$. Since $\mu_E = \sum_{i=0}^{d-1} \frac{r_i}{r}\delta_{\mu_i}$, the equality (14) holds. □

By the above lemma, one has

$$\begin{aligned}\mathrm{vol}(L) &= \dim(X)\,\mathrm{vol}(L_K) \lim_{n\to\infty} \frac{1}{n} \int x_+ \nu_{\pi_*(L^{\otimes n})}(\mathrm{d}x)\\ &= \dim(X)\,\mathrm{vol}(L_K) \lim_{n\to\infty} \int x_+ T_{\frac{1}{n}} \nu_{\pi_*(L^{\otimes n})}(\mathrm{d}x).\end{aligned}$$

From the first part of the theorem, the sequence of measures $(T_{\frac{1}{n}}\nu_{\pi_*(L^{\otimes n})})_{n\geqslant 1}$ converges vaguely to $\nu_L^\pi$. Furthermore, [5] Proposition 4.1 1) shows that the supports of measures $(T_{\frac{1}{n}}\nu_{\pi_*(L^{\otimes n})})_{n\geqslant 1}$ are uniformly bounded from above. Therefore we obtain

$$\mathrm{vol}(L) = \dim(X)\,\mathrm{vol}(L_K) \int x_+ \nu_L^\pi(\mathrm{d}x),$$

which proves the theorem.

***Remark 2.6.*** — 1) By definition, for any integer $n \geqslant 1$, one has $\nu^\pi_{L^{\otimes n}} = T_n \nu_L^\pi$. Hence we recover the equality $\mathrm{vol}(L) = n^{\dim X}\,\mathrm{vol}(L)$.

2) Let $M$ be a line bundle of degree $a$ on $C$, and $E$ is a non-zero vector bundle on $C$ whose Harder-Narasimhan filtration is as in (2) and whose successive slopes are as in (3). Assume that $M$ is a line bundle of degree $a$ on $E$. Then the Harder-Narasimhan flag of $E\otimes M$ is just $E\otimes M = E_0\otimes M \supsetneq E_1\otimes M \supsetneq \cdots \supsetneq E_d\otimes M = 0$, and its successive slopes are $\mu_i + a$, $i \in \{0, \cdots, d-1\}$. Therefore, one has $\nu_{E\otimes M} = \tau_a\nu_E$, where for any Borel probability measure $\nu$ on $\mathbb{R}$, $\tau_a\nu$ is defined as

$$\int f(x)\tau_a\nu(\mathrm{d}x) = \int f(x+a)\nu(\mathrm{d}x).$$

One verifies that $T_\varepsilon\tau_a\nu = \tau_{a\varepsilon}T_\varepsilon\nu$. Thus we obtain that, for any integer $n \geqslant 1$, $\pi_*((L\otimes\pi^*M)^{\otimes n}) = \pi_*(L^{\otimes n}) \otimes M^{\otimes n}$. Therefore, one has $\nu_{\pi_*((L\otimes\pi^*M)^{\otimes n})} = \tau_{na}\nu_{\pi_*(L^{\otimes n})}$, which implies that $\nu^\pi_{L\otimes\pi^*M} = \tau_a\nu^\pi_L$. In particular,

$$\mathrm{vol}(L\otimes\pi^*M) = \dim(X)\,\mathrm{vol}(L_K) \int (x+a)_+ \nu_L^\pi(\mathrm{d}x).$$

## 3. Harder-Narasimhan filtrations of tensor powers

Let $E$ be a non-zero vector bundle on $C$ whose Harder-Narasimhan filtration is

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0 \tag{15}$$

and whose successive slopes are $\mu_i = \mu(E_i/E_{i+1})$. For any $i \in \{0, \cdots, d\}$, let $\mathrm{sq}_i(E)$ be the sub-quotient $E_i/E_{i+1}$. In this section, we shall determine, for any integer $n \geqslant 1$, the Harder-Narasimhan filtration of $E^{\otimes n}$ and that of $S^n E$. Here we should suppose that the characteristic of the ground field $k$ is zero, so that the following results of Ramanan and Ramanathan [**13**] hold:

1) The tensor product of two semistable vector bundles on $C$ is still semistable.
2) Any symmetric power of a semistable vector bundle $E$ on $C$ is semistable. Furthermore, one has $\mu(S^n E) = n\mu(E)$.
3) If $E_1$ and $E_2$ are two non-zero vector bundles on $C$, then

$$\mu_{\max}(E_1 \otimes E_2) = \mu_{\max}(E_1) + \mu_{\max}(E_2), \quad \mu_{\min}(E_1 \otimes E_2) = \mu_{\min}(E_1) + \mu_{\min}(E_2).$$

**3.1. The case of $E^{\otimes n}$.** — Denote by $\Theta$ the indices set $\{0, \cdots, d-1\}^n$. We introduce a partial order "$\leqslant$" on $\Theta$ such that

$$(j_1, \cdots, j_n) \leqslant (l_1, \cdots, l_n) \text{ if and only if } j_1 \leqslant l_1, \cdots, j_n \leqslant l_n.$$

We say that a subset $A$ of $\Theta$ is *saturated* if

$$\alpha \in A,\ \beta \in \Theta \text{ and } \beta \geqslant \alpha \implies \beta \in A.$$

Assume that $A$ is an arbitrary subset of $\Theta$. Then $\{\beta \mid \exists\, \alpha \in A,\ \beta \geqslant \alpha\}$ is the smallest saturated subset of $\Theta$ containing $A$. We denote it by $\overline{A}$.

For any $\alpha = (a_1, \cdots, a_n) \in \Theta$, denote by $E_\alpha$ the tensor product $E_{a_1} \otimes \cdots \otimes E_{a_n}$ and by $\mathrm{sq}_\alpha(E)$ the tensor product of sub-quotients $\mathrm{sq}_{a_1}(E) \otimes \cdots \otimes \mathrm{sq}_{a_n}(E)$. Since each sub-quotient $\mathrm{sq}_i(E)$ is semistable, also is the tensor product $\mathrm{sq}_\alpha(E)$. For any non-empty subset $A$ of $\Theta$, let $E_A := \sum_{\alpha \in A} E_\alpha$. Write by convention $E_\varnothing = 0$. In the following are some basic properties of vector bundles $E_A$:

1) $E_A = E_{\overline{A}}$.
2) If $A_1 \subset A_2 \subset \Theta$, then $E_{A_1} \subset E_{A_2}$.

***Proposition 3.1.*** — *Assume that $A \subset \Theta$ is non-empty and saturated, and that $A''$ is a subset of $A$ consisting of maximal elements. Then $A' := A \setminus A''$ is also saturated. Furthermore, one has an isomorphism*

$$E_A/E_{A'} \cong \bigoplus_{\alpha \in A''} \mathrm{sq}_\alpha(E). \tag{16}$$

*Proof.* — Locally for the Zariski topology, the flag (15) is split. Therefore a classical argument in linear algebra leads locally to a natural isomorphism of the form (16). These natural isomorphisms do not depend on the choice of the splitting and therefore glue together into a global isomorphism. □

For any $\alpha = (a_1, \cdots, a_n) \in \Theta$, let $\mu_\alpha$ be the sum $\mu_{a_1} + \cdots + \mu_{a_n}$. With this notation, the slope of $\mathrm{sq}_\alpha(E)$ is just $\mu_\alpha$. Denote by $\Sigma_n$ the set of real numbers of the form $\mu_\alpha$ where $\alpha$ takes through vectors in $\Theta$. From the inequalities $\mu_0 < \mu_1 < \cdots < \mu_{d-1}$ we obtain that $\alpha \leqslant \beta$ implies $\mu_\alpha \leqslant \mu_\beta$, and $\alpha \lneqq \beta$ implies $\mu_\alpha < \mu_\beta$.

***Proposition 3.2.*** — *The set $\Sigma_n$ identifies with that of successive slopes of $E^{\otimes n}$. Furthermore, suppose that the elements in $\Sigma_n$ are ordered as*

$$v_0 < v_1 < \cdots < v_{m-1},$$

*then $E_{A_0} \supsetneq E_{A_1} \supsetneq \cdots \supsetneq E_{A_m} = 0$ is the Harder-Narasimhan filtration of $E^{\otimes n}$, where $A_j = \{\alpha \mid \mu_\alpha \geqslant v_j\}$ for $j \in \{0, \cdots, m-1\}$, and $A_m = \varnothing$.*

*Proof.* — By the definition of Harder-Narasimhan filtration, it suffices to prove that, for each $j \in \{0, \cdots, m-1\}$, the sub-quotient $E_{A_j}/E_{A_{j+1}}$ is semistable of slope $v_j$. We have $A_j \setminus A_{j+1} = \{\alpha \,|\, \mu_\alpha = v_j\}$. Moreover, if $\mu_\alpha = v_j$, then $\alpha$ is a maximal element of $A_j$. Therefore, Proposition 3.1 implies that

$$E_{A_j}/E_{A_{j+1}} \cong \bigoplus_{\alpha \in A_j \setminus A_{j+1}} \mathrm{sq}_\alpha(E)$$

is semi-stable of slope $v_j$. □

***Remark 3.3.*** — In **[3]**, the author has introduced another interpretation of Harder-Narasimhan filtration: one defines, for any $t \in \mathbb{R}$,

$$\mathcal{F}_t^{\mathrm{HN}} := \sum_{\substack{0 \neq F \subset E \\ \mu_{\min}(F) \geqslant t}} F. \tag{17}$$

This is a decreasing $\mathbb{R}$-filtration of $E$. The subbundles of $E$ appearing in the $\mathbb{R}$-filtration (17) are just $E_i$, $i \in \{0, \cdots, d\}$. Furthermore, the measure $\nu_E$ defined in (4) coincides with the first order derivative (in the sense of distribution) of the function $t \mapsto -\operatorname{rk} \mathcal{F}_t^{\mathrm{HN}} E / \operatorname{rk} E$. Proposition 3.2 shows that the $\mathbb{R}$-filtration of $E^{\otimes n}$ is just the $n^{\mathrm{th}}$ tensor power of the $\mathbb{R}$-filtration of $E$. More precisely, one has

$$\mathcal{F}_t^{\mathrm{HN}} E^{\otimes n} = \sum_{t_1 + \cdots + t_n = t} \mathcal{F}_{t_1}^{\mathrm{HN}} E \otimes \cdots \otimes \mathcal{F}_{t_n}^{\mathrm{HN}} E, \tag{18}$$

and therefore $\nu_{E^{\otimes n}} = \nu_E^{*n}$, where $*$ denotes the convolution of Borel probability measures on $\mathbb{R}$.

**3.2. The case of $S^n E$.** — The symmetric group $\mathfrak{S}_n$ acts on $E^{\otimes n}$ by permuting factors. The $n^{\mathrm{th}}$ symmetric power of $E$ is defined as the quotient of $E^{\otimes n}$ by the action of $\mathfrak{S}_n$. Let us keep the notation of § 3.1. The group $\mathfrak{S}_n$ acts naturally on $\Theta = \{0, \cdots, d-1\}^n$. Denote by $\widetilde{\Theta}$ the quotient space $\Theta/\mathfrak{S}_n$. Each class $[\alpha]$ in $\widetilde{\Theta}$ corresponds bijectively to a partition of $n$ into sum of $d$ non-negative integers $(a_i)_{i=0}^{d-1}$. For such a class $[\alpha]$, we use the symbol $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ to denote the tensor product $\bigotimes_{i=0}^{d-1} S^{a_i} \mathrm{sq}_i(E)$. One observes that the group $\mathfrak{S}_n$ acts naturally on $\bigoplus_{\beta \in [\alpha]} \mathrm{sq}_\beta(E)$ and its quotient space by the action of $\mathfrak{S}_n$ is just $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$. Moreover, $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ is also semi-stable of slope $\mu_\alpha$.

For any $A \subset \Theta$ which is invariant by the action of $\mathfrak{S}_n$, denote by $S_A E$ the image of $E_A$ in $S^n E$. It is the quotient of $E_A$ by the action of $\mathfrak{S}_n$. Let $A_j \subset \Theta$, $j \in \{0, \cdots, m\}$ be as in Proposition 3.2. By the uniqueness of Harder-Narasimhan filtration, each $A_j$ is invariant by the action of $\mathfrak{S}_n$.

***Proposition 3.4.*** — *The set of successive slopes of $S^n E$ is also $\Sigma_n$. Furthermore, $S_{A_0}E \supsetneq S_{A_1}E \supsetneq \cdots \supsetneq S_{A_m}E = 0$ is the Harder-Narasimhan filtration of $S^n E$.*

*Proof.* — As pointed out above, each vector bundle $E_{A_j}$ is invariant by the action of $\mathfrak{S}_n$, and $S_{A_j}E$ is the quotient. Therefore, we obtain

$$S_{A_j}E/S_{A_{j+1}}E \cong \bigoplus_{[\alpha]\in(A_j\setminus A_{j+1})/\mathfrak{S}_n} \widetilde{\mathrm{sq}}_{[\alpha]}(E).$$

Since $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ is semi-stable of slope $\mu_\alpha$, $S_{A_j}E/S_{A_{j+1}}E$ is semi-stable of slope $v_j$. The proposition is then proved. □

In the following, we compute the measure $\nu_{S^n E}$. Take an arbitrary class $[\alpha]$ in $\widetilde{\Theta}$ which corresponds to the partition $\underline{a} = (a_i)_{i=0}^{d-1}$ of $n$. The value $\mu_\alpha$ (which does not depend on the representing element $\alpha$) equals $\sum_{i=0}^{d-1} a_i \mu_i$. For each $i \in \{0, \cdots, d-1\}$, let $r_i$ be the rank of $\mathrm{sq}_i(E)$. The rank of $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ is just

$$r_{\underline{a}} := \prod_{i=0}^{d-1} \binom{a_i + r_i - 1}{r_i - 1}.$$

We then obtain that

$$R_j := \mathrm{rk}(S_{A_j}E/S_{A_{j+1}}E) = \sum_{\substack{\underline{a}=(a_i)_{i=0}^{d-1}\in\mathbb{N}^d \\ a_0+\cdots+a_{d-1}=n \\ a_0\mu_0+\cdots+a_{d-1}\mu_{d-1}=v_j}} r_{\underline{a}}.$$

By definition, one has

$$\nu_{S^n E} = \frac{1}{\mathrm{rk}\, S^n E} \sum_{j=0}^{m-1} R_j \delta_{v_j}, \tag{19}$$

where $\delta_{v_j}$ is the Dirac measure concentrated at $v_j$.

The formula (19) gives an explicit description of the probability measure $\nu_{S^n E}$. However, it seems that in the case where the $r_i$ ($i \in \{0, \cdots, d-1\}$) are different, the values of $R_j$, $j \in \{0, \cdots, m-1\}$ are rather tedious to calculate. In the following we propose another explicit form of $\nu_{S^n E}$ which is adapted to the proof of Theorem 1.2.

***Proposition 3.5.*** — *Let $r = r_0 + \cdots + r_{d-1}$ and $\underline{s} = (s_1, \cdots, s_r)$ be a vector in $\mathbb{R}^n$ such that the value $\mu_i$ appears exactly $r_i$ times in the coordinates on $\underline{s}$. Then*

$$\nu_{S^n E} = \frac{1}{\mathrm{rk}\, S^n E} \sum_{\substack{\underline{b}=(b_l)_{l=1}^{r}\in\mathbb{N}^r \\ b_1+\cdots+b_r=n}} \delta_{b_1 s_1+\cdots+b_r s_r}. \tag{20}$$

*Proof.* — The only thing to verify is that, for any $v \in \mathbb{R}$,

$$\sum_{\substack{\underline{a}=(a_i)_{i=0}^{d-1}\in\mathbb{N}^d\\ a_0+\cdots+a_{d-1}=n\\ a_0\mu_0+\cdots+a_{d-1}\mu_{d-1}=v}} r_{\underline{a}} = \sum_{\substack{\underline{b}=(b_l)_{l=1}^{r}\in\mathbb{N}^r\\ b_1+\cdots+b_r=n\\ b_1s_1+\cdots+b_rs_r=v}} 1.$$

This relies on the fact that $\binom{a_i+r_i-1}{r_i-1}$ equals the number of partitions of $a_i$ into sum of $r_i$ positive integers. □

**3.3. Proofs of Theorem 1.2 and Corollary 1.3.** — By definition, $\nu^{\pi}_{\mathcal{O}_E(1)}$ coincides with the vague limit of measures $T_{\frac{1}{n}}\nu_{S^nE}$ when $n \to \infty$. By Proposition 3.5,

$$\begin{aligned} T_{\frac{1}{n}}\nu_{S^nE} &= \frac{1}{\operatorname{rk} S^nE}\sum_{\substack{\underline{b}=(b_l)_{l=1}^{r}\in\frac{1}{n}\mathbb{N}^r\\ b_1+\cdots+b_r=1}} \delta_{b_1d_1+\cdots+b_rd_r}\\ &= \varphi_{\underline{s}*}\Big[\frac{1}{\operatorname{rk} S^nE}\sum_{\underline{b}\in\frac{1}{n}\mathbb{N}^r\cap\Delta}\delta_{\underline{b}}\Big], \end{aligned}$$

where in the last bracket is the measure of the $n^{\text{th}}$ Riemann sum on $\Delta$. Therefore, $(T_{\frac{1}{n}}\nu_{S^nE})_{n\geqslant 1}$ converges vaguely to $\varphi_{\underline{s}*}\eta$. Theorem 1.2 is thus proved.

Finally, by remark 2.6, one has

$$\begin{aligned} \operatorname{vol}(\mathcal{O}_E(m)\otimes\pi^*M) &= r\operatorname{vol}(\mathcal{O}_E(m)_K)\int(x+c)_+\nu_{\mathcal{O}_E(m)^\pi}(\mathrm{d}x)\\ &= rm^{r-1}\int(mx+c)_+\nu^{\pi}_{\mathcal{O}_E(1)}(\mathrm{d}x)\\ &= rm^r\int(x+c/m)_+\varphi_{\underline{s}*}\eta(\mathrm{d}x), \end{aligned}$$

which proves Corollary 1.3.

## 4. Polynomial representation of the volume function

In this section, we show that Corollary 1.3 permits to get the subdivision of the Néron-Severi group $N^1(\mathbb{P}(E))$ into sectors such that the volume function is polynomial on each sector. We shall keep the notation introduced in Theorem 1.2 and Corollary 1.3. Note that the value in (6) depends only on $m$ and $c$, we denote it by $\Phi(m,c)$.

Firstly, let us discuss some degenerate cases. Assume that $E$ is semistable, i.e., $d=1$. Then the limit measure $\nu^{\pi}_{\mathcal{O}(E)}$ is the Dirac measure concentrated at $\mu(E)$. Therefore, Corollary 1.3 implies

$$\Phi(m,c) = rm^r(\mu(E)+c/m)_+ = \begin{cases} \deg(E)m^r + crm^{r-1}, & \text{if } c \geqslant -\mu(E)m,\\ 0, & \text{else.}\end{cases}$$

If $r = d = 2$, then the measure $\varphi_{\underline{s}*}\eta$ is the uniform distribution on the interval $[\mu_0, \mu_1]$. In this case

$$\Phi(m,c) = \frac{2m^2}{\mu_1 - \mu_0}\int_{\mu_0}^{\mu_1}(x+c/m)_+\,\mathrm{d}x = \begin{cases} m^2(\mu_1+\mu_0)+2cm, & c > -m\mu_0, \\ (m\mu_1+c)^2/(\mu_1-\mu_0), & -m\mu_1 \leqslant c \leqslant m\mu_0, \\ 0, & c < -m\mu_1. \end{cases}.$$

In the following, we always suppose $r \geqslant 3$ and $d \geqslant 2$. Without loss of generality, we assume that the coordinates of $\underline{s}$ are ordered such that $s_1 \leqslant \cdots \leqslant s_r$. For any $i \in \{0, \cdots, d-1\}$, let $j_i = \mathrm{rk}(E/E_{i+1})$. Let $j_{-1} = 0$. Note that $s_l = \mu_i$ if and only if $l \in ]j_{i-1}, j_i]$. For any real number $t \in [d_1, d_r]$, let $\Delta_t$ be the intersection of $\Delta$ and the hyperplane $\{(x_1, \cdots, x_r) \mid s_1x_1 + \cdots + s_rx_r = t\}$. Denote by $v_t$ the density of $\varphi_{\underline{s}*}\eta$ with respect to the Lebesgue measure. Since for any $i \in \{1, \cdots, d-1\}$, the polytopes $\Delta_t$ ($t \in [\mu_{i-1}, \mu_i]$) are similar, the function $v_t$ is a polynomial of degree $r-2$ on $[\mu_{i-1}, \mu_i]$. Hence there exists a polynomial $P_i$ of degree $r-2$ such that $v_t = P_i(t)$ on $[\mu_{i-1}, \mu_i]$. Thus we obtain

$$\Phi(m,c) = rm^r\sum_{i=1}^{d-1}\int_{\mu_{i-1}}^{\mu_i}(t+c/m)_+P_i(t)\,\mathrm{d}t.$$

Therefore, $\Phi(m,c) = 0$ if $c \leqslant -\mu_{d-1}m$. If $c \in [-\mu_j m, -\mu_{j-1}m]$ ($j \in \{1, \cdots, d-1\}$), one has

$$\begin{aligned}\Phi(m,c) = r\sum_{i=j+1}^{d-1}&\left[\left(\int_{\mu_{i-1}}^{\mu_i} tP_i(t)\,\mathrm{d}t\right)m^r + \left(\int_{\mu_{i-1}}^{\mu_i} P_i(t)\,\mathrm{d}t\right)cm^{r-1}\right] \\ &+ r\left[\left(\int_{-c/m}^{\mu_j} tP_i(t)\,\mathrm{d}t\right)m^r + \left(\int_{-c/m}^{\mu_i} P_i(t)\,\mathrm{d}t\right)cm^{r-1}\right],\end{aligned}$$

which is a homogeneous polynomial of degree $r$ in $m$ and $c$. Finally, when $c > -\mu_0 m$, then

$$\Phi(m,c) = r\sum_{i=1}^{d-1}\left[\left(\int_{\mu_{i-1}}^{\mu_i} tP_i(t)\,\mathrm{d}t\right)m^r + \left(\int_{\mu_{i-1}}^{\mu_i} P_i(t)\,\mathrm{d}t\right)cm^{r-1}\right]$$

has degree 1 in $c$.

## References


[1] S. Boucksom, J.-P. Demailly, M. Paun & T. Peternell – "The pseudo-effective cone of a compact Käler manifold and varieties of negative Kodaira dimension", preprint, 2004.

[2] S. Boucksom, C. Favre & M. Jonsson – "Differentiability of volumes of divisors and a problem of Teissier", to appear in *Journal of algebraic geometry*, 2006.

[3] H. Chen – "Convergence des polygones de Harder-Narasimhan", to appear in *Mémoire de la Société Mathématique de France*, 2007.

[4] ———, "Arithmetic Fujita approximation", arXiv:0810.5479, 2008.

[5] ———, "Positive degree and arithmetic bigness", arXiv:0803.2583, 2008.

[6] S. D. CUTKOSKY & V. SRINIVAS – "On a problem of Zariski on dimensions of linear systems", *Annals of Mathematics. Second Series* **137** (1993), no. 3, p. 531–559.

[7] L. EIN, R. LAZARSFELD, M. MUSTAŢĂ, M. NAKAMAYE & M. POPA – "Asymptotic invariants of line bundles", *Pure and Applied Mathematics Quarterly* **1** (2005), no. 2, p. 379–403.

[8] ———, "Asymptotic invariants of base loci", *Université de Grenoble, Annals de l'Institut Fourier* **56** (2006), no. 6, p. 1701–1734.

[9] T. FUJITA – "Approximating Zariski decomposition of big line bundles", *Kodai Mathematical Journal* **17** (1994), no. 1, p. 1–3.

[10] R. LAZARSFELD – *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[11] R. LAZARSFELD & M. MUSTAŢĂ – "Convex bodies associated to linear series", preprint, 2008.

[12] N. NAKAYAMA – *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[13] S. RAMANAN & A. RAMANATHAN – "Some remarks on the instability flag", *The Tohoku Mathematical Journal. Second Series* **36** (1984), no. 2, p. 269–291.

[14] S. TAKAGI – "Fujita's approximation theorem in positive characteristics", *Journal of Mathematics of Kyoto University* **47** (2007), no. 1, p. 179–202.

[15] A. WOLFE – "Asymptotic invariants of graded systems of ideals and linear systems on projective bundles", Ph.D. Thesis, University of Michigan, 2005.

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