COMPARISON OF SOME INVARIANTS OF EUCLIDEAN LATTICES

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1. Introduction

The study of Euclidean lattices has a long history and has many applications in diverse branches of mathematics such as number theory, Lie theory, convex geometry, cryptography etc. By definition a Euclidean lattice is a discrete subgroup of a Euclidean space which has maximal rank over $\mathbb{Z}$. Equivalently, we can also view a Euclidean lattice as a free abelian group of finite rank E, equipped with a Euclidean norm on $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R}$. Among the invariants of Euclidean lattices, the successive minima of Minkowski have a central interest in geometry of numbers. Given a Euclidean lattice $(E, \|\cdot\|)$ of rank r, the i^{th} minimum of $(E, \|\cdot\|)$ is defined as the smallest $r > 0$ such that the lattices points of length $\leqslant r$ span a vector subspace of $E_{\mathbb{R}}$ whose rank is at least i. In order to facilite the presentation of the article, we denote by $\widehat{\nu}_i(E, \|\cdot\|)$ the opposite logarithmic version of the i^{th} minimum, defined as

$$\widehat{\nu}_i(E, \|\cdot\|) := -\ln \inf\{r > 0 \,:\, \mathrm{rk}(\{x \in E \,:\, \|x\| \leqslant r\}) \geqslant i\}.$$

The family

$$\mathrm{span}_{\mathbb{Z}}(\{x \in E \,:\, \|x\| \leqslant r\}), \quad r \geqslant 0$$

form a flag of sub-$\mathbb{Z}$-modules of E, which is called the *Minkowski reduction* of the Euclidean lattice $(E, \|\cdot\|)$.

Motivated by the similarity of arithmetic properties of number fields and function fields, Stuhler [**25**] has considered Euclidean lattices as the analogue of vector bundles over a smooth projective curve and developed a Harder-Narasimhan theory in the arithmetic setting. The starting point is the Arakelov degree function. Let $(E, \|\cdot\|)$ be a Euclidean lattice. The Arakelov degree of $(E, \|\cdot\|)$ is defined as

$$\widehat{\deg}(E, \|\cdot\|) := -\ln\|e_1 \wedge \cdots \wedge e_n\|_{\det}, \tag{1.1}$$

where $\{e_i\}_{i=1}^n$ is a basis of E over $\mathbb{Z}$, and $\|\cdot\|_{\det}$ is the determinant norm on $\det(E_{\mathbb{R}})$ associated with $\|\cdot\|$, defined as

$$\forall\, \eta \in \det(E_{\mathbb{R}}), \quad \|\eta\|_{\det} := \inf_{\substack{(x_1,\ldots,x_n)\in E^n \\ \eta = x_1\wedge\cdots\wedge x_n}} \|x_1\|\cdots\|x_n\|.$$

If $\{e_i\}_{i=1}^n$ is a basis of E over $\mathbb{Z}$, then $\|e_1 \wedge \cdots \wedge e_n\|_{\det}$ identifies with the volume of the fundamental domain

$$\{t_1e_1 + \cdots + t_ne_n \,:\, (t_1, \ldots, t_n) \in [0, 1]^n\}.$$

Therefore the value of (1.1) does not depend on the choice of the basis of E over $\mathbb{Z}$. It is actually the opposite logarithmic version of the covolume of the lattice in classic geometry of numbers. Arakelov degree is also similar to the notion of degree in the setting of vector bundles over a smooth projective curve. In particular, if

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is a short exact sequence of free abelian groups of finite rank, $\|\cdot\|$ is a Euclidean norm on E, $\|\cdot\|'$ is the restriction of $\|\cdot\|$ on $E'_{\mathbb{R}}$, and $\|\cdot\|''$ is the quotient norm of $\|\cdot\|$ on $E''_{\mathbb{R}}$, then the following equality holds

$$\widehat{\deg}(E, \|\cdot\|) = \widehat{\deg}(E', \|\cdot\|') + \widehat{\deg}(E'', \|\cdot\|''). \tag{1.2}$$

Moreover, it can be shown that, if E_1 and E_2 are two subgroups of E, then the following inequality holds

$$\widehat{\deg}(\overline{E_1 \cap E_2}) + \widehat{\deg}(\overline{E_1 + E_2}) \geqslant \widehat{\deg}(\overline{E_1}) + \widehat{\deg}(\overline{E_2}), \tag{1.3}$$

where for any subgroup F of E, the expression $\overline{F}$ denotes the Euclidean lattice consisting of F and the restriction of $\|\cdot\|$ on $F_{\mathbb{R}}$.

Assume that the Euclidean lattice $\overline{E} = (E, \|\cdot\|)$ is non-zero, the *slope* of $(\overline{E})$ is defined as

$$\widehat{\mu}(\overline{E}) := \frac{\widehat{\deg}(\overline{E})}{\mathrm{rk}_{\mathbb{Z}}(E)}.$$

Let $\widehat{\mu}_{\max}(\overline{E})$ be

$$\sup_{\{0\}\neq F\subset E}\widehat{\mu}(\overline{F}),$$

where F runs over the set of all non-zero subgroups of E. The quantity $\widehat{\mu}_{\max}(\overline{E})$ is called the *maximal slope* of $\overline{E}$. If for any subgroup F of E, $\widehat{\mu}(\overline{F})$ is bounded from above by $\widehat{\mu}(\overline{E})$, or equivalently $\widehat{\mu}(\overline{E})=\widehat{\mu}_{\max}(\overline{E})$, we say that the Euclidean lattice $\overline{E}=(E,\|\cdot\|)$ is *semi-stable*. By using the relations (1.2) and (1.3), Stuhler has proved that there exists a unique subgroup E_{des} of E such that

$$\widehat{\mu}(\overline{E_{\mathrm{des}}})=\widehat{\mu}_{\max}(\overline{E})$$

and which contains all subgroups F satisfying $\widehat{\mu}(\overline{F})=\widehat{\mu}_{\max}(\overline{E})$. The subgroup E_{des} equipped with the restriction of $\|\cdot\|$ is called the *destabilising sublattice* of the Euclidean lattice $\overline{E}$. This construction allows to obtain a flag of subgroups of E

$$0=E_0\subsetneq E_1\subsetneq\ldots\subsetneq E_d=E \tag{1.4}$$

such that $\overline{E_i/E_{i-1}}$ is the destabilising sublattice of $\overline{E/E_{i-1}}$. It is also the unique flag of subgroups of E such that each subquotient E_i/E_{i-1} is free and that the following inequalities hold

$$\widehat{\mu}(\overline{E}_1)>\ldots>\widehat{\mu}(\overline{E_d/E_{d-1}}).$$

This is an arithmetic analogue of a result of Harder and Narasimhan (see **[20]** Lemma 1.3.7). For this reason, the flag (1.4) is called the *Harder-Narasimhan filtration* of $\overline{E}$. The construction above allows to define another series of arithmetic invariants of Euclidean lattices, called *successive slopes*. Let $\overline{E}=(E,\|\cdot\|)$ be a non-zero Euclidean lattice and n be the rank of E over $\mathbb{Z}$. Let

$$0=E_0\subsetneq E_1\subsetneq\ldots\subsetneq E_d=E$$

be the Harder-Narasimhan filtration of $\overline{E}$. We denote by $\{\widehat{\mu}_i(\overline{E})\}_{i=1}^n$ the decreasing sequence of real numbers, in which $\widehat{\mu}(\overline{E_j/E_{j-1}})$ appears exactly $\mathrm{rk}_{\mathbb{Z}}(E_j)-\mathrm{rk}_{\mathbb{Z}}(E_{j-1})$ times. One can also interpret $\widehat{\mu}_i(\overline{E})$ as the slope on the interval $[i,i+1]$ of the Harder-Narasimhan polygone of $(E,\|\cdot\|)$, which is the piecewise affine function on $[0,\mathrm{rk}_{\mathbb{Z}}(E)]$ whose graph is the upper boundary of the convex envelop of points of coordinates $(\mathrm{rk}_{\mathbb{Z}}(F),\|\cdot\|_F)$, where F is a subgroup of E and $\|\cdot\|_F$ is the restriction of $\|\cdot\|$ on $F_{\mathbb{R}}$. Later this theory has been developed by Grayson **[18]** in the setting of lattices over an algebraic integer ring. Inspired by these results, Bost **[4]** has proposed a slope method in the setting of Hermitian vector bundles over an arithmetic curve, and has used it as a tool to interpret the approach of Masser and Wüstholz **[22]** to Faltings' finiteness theorem **[12, 13]** on the set of isomorphism class of abelian varieties over a number field which are isogenous to a given abelian variety. The slope method has also been applied in **[5]** (see also the survey lecture **[6]**) to study the algebraicity of formal schemes over a number field and applications to several problems of Diophantine geometry, including the conjecture of Grothendieck-Katz [**21**]. The slope theory has been generalised by Gaudron [**14**] to the setting of adelic vector bundles over a global field, in taking into account non-Hermitian norms over Archimedean places, and non-algebraic norms over finite places. More generally, as shown in [**11**, §4.3], an analogue of Harder-Narasimhan theory also holds for adelic vector bundles over an M-field in the sense of Gubler [**19**].

Compared to Minkowski's successive minima, the successive slopes have several advantages in view of applications to Diophantine geometry. First, the successive slopes remain unchanged by a finite extension of the number field (see [**4**] Proposition A.2). Second, the successive slopes behave better by passing to dual. More precisely, if $(E,\|\cdot\|)$ is a non-zero Euclidean lattice of rank $n\in\mathbb{N}_{>0}$, for any $i\in\{1,\ldots,n\}$ one has

$$\widehat{\mu}_i(E,\|\cdot\|)+\widehat{\mu}_{n+1-i}(E^\vee,\|\cdot\|_*)=0, \tag{1.5}$$

where $\|\cdot\|_*$ denotes the dual norm of $\|\cdot\|$ on $E^\vee_{\mathbb{R}}\cong(E_{\mathbb{R}})^\vee$. If we replace successive slopes by successive minima in the above formula, in general the equality does not hold and we only have an inequality of the form

$$\widehat{\nu}_i(E,\|\cdot\|)+\widehat{\nu}_{n+1-i}(E^\vee,\|\cdot\|_*)\leqslant 0.$$

The lower bound of $\widehat{\nu}_i(E,\|\cdot\|)+\widehat{\nu}_{n+1-i}(E^\vee,\|\cdot\|_*)$ is a deep problem in geometry of numbers, known as *transference problem*. Third, by the additivity of the Arakelov degree function (1.2), for any non-zero Euclidean lattice $\overline{E}$ of rank $n\in\mathbb{N}_{>0}$, the following equality holds

$$\widehat{\deg}(\overline{E})=\sum_{i=1}^n\widehat{\mu}_i(\overline{E}).$$

If we replace the successive slopes by successive minima, the equality does not hold in general. It is however possible to estimate the difference between the Arakelov degree (which is the sum of successive slopes) and the sum of successive minima: Minkowski's second theorem can be interpreted as

$$0\leqslant\widehat{\deg}(\overline{E})-\sum_{i=1}^n\widehat{\nu}_i(\overline{E})\leqslant n\ln(2)-\ln(v_n), \tag{1.6}$$

where v_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

In the works mentioned above, the slopes play the role of Minkowski's minima in the geometrisation of Diophantine approximation. Hence the comparison of these series of invariants becomes an interesting problem. This problem is not only important in the philosophic aspect to confirm the adequacy of the slope method to Diophantine problems, but also useful in the explicit estimation of error terms appearing in a Diophantine argument. In the literature, Soulé [**24**, §1.2.5] has firstly pointed out, without providing concrete estimates, that given a Euclidean lattice $\overline{E}$, for any $i \in \{1,\ldots,\mathrm{rk}_{\mathbb{Z}}(E)\}$, $\mu_i(\overline{E})$ and $\nu_i(\overline{E})$ should be close. However, compared to Minkowski's second theorem (1.6), which could be considered as the comparison between the averages of slopes and minima, the comparison between each slope and the corresponding Minkowski's minimum turns out to be much harder. The first explicit comparison between successive slopes and minima has been given by Borek [**2**] in the setting of Hermitian vector bundles over an arithmetic curve. For the sake of simplicity, we recall his result for Euclidean lattices. Let $\overline{E}$ be a Euclidean lattice of rank $n \in \mathbb{N}_{>0}$. Borek has shown that, for any $i \in \{1,\ldots,n\}$, one has

$$0 \leqslant \widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i(\overline{E}) \leqslant i\Big(\ln(2) - \frac{1}{n}\ln(v_n)\Big), \tag{1.7}$$

where v_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. His method consists in applying Minkowski's second theorem to sub-lattices of $(E,\|\cdot\|)$. Note that the case where $i = 1$ could be considered as an interpretation of Minkowski's first theorem. However, in view of (1.6), which can be rewritten as

$$0 \leqslant \sum_{i=1}^{n}\big(\widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i(\overline{E})\big) \leqslant n\ln(2) - \ln(v_n),$$

the upper bound in (1.7) seems to be too weak when i is large.

In [**9**], a uniform upper bound for the difference between slopes and minima has been announced and the details of proof were given in [**10**]. The main idea is to show that the difference between the last slope and the last minimum dominates that of other slopes and corresponding minima. More precisely, we have the following result (see [**10**, Theorem 3.7], see also [**16**, §4.2]).

Theorem 1.1. — *Let $\delta : \mathbb{N}_{>0} \to \mathbb{R}_{\geqslant 0}$ be the map sending any $n \in \mathbb{N}_{>0}$ to*

$$\sup\{\widehat{\mu}_n(\overline{F}) - \widehat{\nu}_n(\overline{F}) \,:\, \overline{F} \text{ is a Euclidean lattice of rank } n\}.$$

Then, for any non-zero Euclidean lattice $\overline{E}$ and any $i \in \{1,\ldots,\mathrm{rk}_{\mathbb{Z}}(E)\}$ one has

$$\widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i(\overline{E}) \leqslant \delta(\mathrm{rk}_{\mathbb{Z}}(E)).$$

Combined with the transference theorem of Banaszczyk [**1**], which implies that

$$\delta(n) \leqslant \ln(n), \tag{1.8}$$

we deduce from Theorem 1.1 the following result. For any non-zero Euclidean lattice $\overline{E}$ and any $i \in \{1,\ldots,\mathrm{rk}_{\mathbb{Z}}(E)\}$, one has (see [**10**, Theorem 1.1])

$$0 \leqslant \widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i(\overline{E}) \leqslant \ln(\mathrm{rk}_{\mathbb{Z}}(E)). \tag{1.9}$$

Note that Stirling's formula implies that

$$\ln(2) - \frac{1}{n}\ln(v_n) \geqslant \frac{\ln(n)}{2} - \frac{1}{2}\ln\Big(\frac{e\pi}{2}\Big) + \frac{1}{2n}\ln(\pi n).$$

Hence the upper bound of (1.9) improves considerably the result of Borek when $i/n \gg 1$, and is better than the upper bound of Borek once $i \geqslant 8$.

The proof of Theorem 1.1 relies on a general principle of comparison between $\mathbb{R}$-filtrations on a finite-dimensional vector space, which can be found in [**9**, Lemma 1.2.1]. The $\mathbb{R}$-filtrations on a finite-dimensional vector space can be identified with ultrametric norms on the vector space, where we consider the trivial absolute value on the underlying field. Let k be a field, equipped with the trivial absolute $|\cdot|_0$, namely

$$\forall\, a \in k, \quad |a|_0 = \begin{cases} 1, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$$

If V is a finite-dimensional real vector space, as *ultrametric norm* on V, we refer to a map $\|\cdot\|_0 : V \to \mathbb{R}_{\geqslant 0}$ which satisfies the following conditions:

(1) for any $a \in k$ and $x \in V$, one has

$$\|ax\|_0 = |a|_0 \cdot \|x\|_0 = \begin{cases} \|x\|, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0, \end{cases}$$

(2) for any $(x, y) \in V \times V$, the strong triangle inequality

$$\|x + y\|_0 \leqslant \max\{\|x\|_0, \|y\|_0\}$$

holds,

(3) for $x \in V$, $\|x\|_0 = 0$ if and only if x is the zero vector.

Such ultrametric norms seem to be very simple. In particular, they induce the discrete topology on the real vector space. However, they actually describe very interesting structures, which are closely related to the arithmetic geometry of Euclidean lattices.

The rest of article is organised as follows. In the second section, we explain the geometry of ultrametrically normed vector spaces over a trivially valued field. In the third section, we give a simplified version of Banaszczyk's proof of the transference theorem. In the fourth and last section, we explain the variants and generalisations of the comparison result and further research topics.

2. Geometry of trivially valued fields

Let k be a field. In this section, we consider the trivial absolute value $|\cdot|_0$ on k, which takes constant value 1 on $k \setminus \{0\}$. If V is a finite-dimensional vector space over k, equipped with an ultrametric norm $\|\cdot\|$, for any $R \geqslant 0$, the central ball of radius R in V, defined as

$$B_R(V, \|\cdot\|) := \{x \in V \,:\, \|x\| \leqslant R\}, \tag{2.1}$$

is a k-vector subspace of V. In fact, the strong triangle inequality shows that $B_R(V, \|\cdot\|)$ is invariant by addition, and the fact that we consider the trivial absolute value on k implies that $B_R(V, \|\cdot\|)$ is stable by multiplication by a scalar in k.

Definition 2.1. — Let $(V, \|\cdot\|)$ be a finite-dimensional vector space over k. For any $i \in \{1, \ldots, \dim_k(V)\}$, let

$$\lambda_i(V, \|\cdot\|) := \sup\{t \in \mathbb{R} \,:\, \mathrm{rk}_k(B_{\mathrm{e}^{-t}}(V, \|\cdot\|)) \geqslant i\}.$$

The sequence

$$\lambda_i(V, \|\cdot\|), \quad i \in \{1, \ldots, \dim_k(V)\}$$

is decreasing. They are analogous to successive minima of Euclidean lattices.

Interestingly, the construction of Harder-Narasimhan filtration is also valid for ultrametrically normed finite-dimensional vector spaces over a trivially valued field. We begin with the definition of the degree function.

Definition 2.2. — Let V be a finite-dimensional vector space over k, equipped with a norm $\|\cdot\|$. Let r be the dimension of V over k. We denote by $\|\cdot\|_{\det}$ the norm on $\det(V) = \Lambda^r(V)$ defined as follows:

$$\forall\, \eta \in \det(V), \quad \|\eta\|_{\det} = \inf_{\substack{(x_1,\ldots,x_r)\in V^r \\ \eta = x_1\wedge\cdots\wedge x_r}} \|x_1\| \cdots \|x_r\|.$$

Note that $\det(V)$ is a one-dimensional vector space over k. Therefore the norm function $\|\cdot\|_{\det}$ is constant on $\det(V) \setminus \{0\}$. We denote by $\deg(V, \|\cdot\|)$ the value

$$-\ln\|\eta\|_{\det},$$

where η is an arbitrary non-zero element of $\det(V)$. If in addition the vector space V is non-zero, we define the *slope* of $(V, \|\cdot\|)$ as

$$\mu(V, \|\cdot\|) := \frac{\deg(V, \|\cdot\|)}{\mathrm{rk}_k(V)}.$$

The *maximal slope* of $(V, \|\cdot\|)$ is defined as

$$\sup_{\{0\}\neq W \subset E} \mu(W, \|\cdot\|_W),$$

where W runs over the set of all non-zero vector subspaces of V, and $\|\cdot\|_W$ denotes the restriction of $\|\cdot\|$ to W.

We consider the analogue of the semi-stability condition in the setting of ultrametrically normed finite-dimensional vector space over a trivially valued field. Let us keep the notation of Definition 2.2 and assume that the vector space V is non-zero. We say that $(V,\|\cdot\|)$ is *semi-stable* if the following equality

$$\mu_{\max}(V,\|\cdot\|)=\mu(V,\|\cdot\|)$$

holds. This condition is equivalent to requiring that the restriction of $\|\cdot\|$ to $V\setminus\{0\}$ is a constant function (see **[11]** Proposition 4.3.61), and the maximal slope of $(V,\|\cdot\|)$ is equal to

$$-\ln\inf_{x\in V\setminus\{0\}}\|x\|.$$

Therefore the smallest non-zero central ball

$$V_{\mathrm{des}}=\big\{x\in V\,:\,\|x\|\leqslant\inf_{y\in V\setminus\{0\}}\|y\|\big\}$$

equipped with the restricted norm is semi-stable of slope $\mu_{\max}(V,\|\cdot\|)$. It also contains all non-zero vector subspaces of V which have $\mu_{\max}(V,\|\cdot\|)$ as their slope. Therefore, the construction of Harder-Narasimhan filtration is also valid, which leads to the existence of a unique sequence

$$0=V_0\subsetneq V_1\subsetneq\ldots\subsetneq V_d=V$$

such that each subquotient V_j/V_{j-1} equipped with the subquotient norm (denoted by $\overline{V_j/V_{j-1}}$) is semi-stable and that

$$\mu(\overline{V_1/V_0})>\ldots>\mu(\overline{V_d/V_{d-1}}).$$

Moreover, for $j\in\{1,\ldots,d\}$, V_j identifies with the central ball of radius $\exp(-\mu(V_j/V_{j-1}))$. In particular, $\{\lambda_i(\overline{V})\}_{i=1}^n$ identifies with the sequence of successive slopes of $\overline{V}$, namely the decreasing sequence of real numbers in which $\mu(V_j/V_{j-1})$ appears exactly $\mathrm{rk}_k(V_j)-\mathrm{rk}_k(V_{j-1})$ times.

Proposition 2.3. — *Let V be a non-zero finite-dimensional vector space over k and $\|\cdot\|_1$ and $\|\cdot\|_2$ be ultrametric norms on V. Let α be the operator norm of the identity map $(V,\|\cdot\|_1)\to(V,\|\cdot\|_2)$, namely,*

$$\alpha:=\sup_{x\in V\setminus\{0\}}\frac{\|x\|_2}{\|x\|_1}.$$

Then, for any $i\in\{1,\ldots,\mathrm{rk}_k(V)\}$, one has

$$\lambda_i(V,\|\cdot\|_1)\leqslant\lambda_i(V,\|\cdot\|_2)+\ln(\alpha).\tag{2.2}$$

Proof. — Let t be a real number such that $\lambda_i(V,\|\cdot\|_1)>t$. By definition the ball $B_{\mathrm{e}^{-t}}(V,\|\cdot\|_1)$ has rank $\geqslant i$ over k. Since α is the operator norm, one has

$$\begin{aligned}B_{\mathrm{e}^{-t}}(V,\|\cdot\|_1)&=\{x\in V\,:\,\|x\|_1\leqslant\mathrm{e}^{-t}\}\\&\subset\{x\in V\,:\,\|x\|_2\leqslant\alpha\mathrm{e}^{-t}\}=B_{\mathrm{e}^{-(t-\ln(\alpha))}}(V,\|\cdot\|_2),\end{aligned}$$

which implies that $B_{\mathrm{e}^{-(t-\ln(\alpha))}}(V,\|\cdot\|_2)$ has rank $\geqslant i$ over k and hence

$$\lambda_i(V,\|\cdot\|_2)\geqslant t-\ln(\alpha).$$

Since $t\in\mathbb{R}$ such that $t<\lambda_i(V,\|\cdot\|_1)$ is arbitrary, the inequality (2.2) holds. □

We have seen that, in the framework of ultrametrically normed finite-dimensional vector spaces over a trivially valued field, the analogues of slopes and minima are the same. We can serve it as a model to understand various arithmetic invariant. Let $\overline{E}=(E,\|\cdot\|)$ be a Euclidean lattice. We equip $\mathbb{R}$ with the trivial absolute value and define a norm $\|\cdot\|_{\widehat{\nu}}$ on $E_{\mathbb{R}}=E\otimes_{\mathbb{Z}}\mathbb{R}$ as follows:

$$\forall\, x\in E_{\mathbb{R}},\quad \|x\|_{\widehat{\nu}}:=\inf\{R\geqslant 0\,:\,x\in\mathrm{Vect}_{\mathbb{R}}(\{y\in E\,:\,\|y\|\leqslant R\})\}.$$

Note that, for any $R\geqslant 0$, the central ball of radius R of $(E_{\mathbb{R}},\|\cdot\|_{\widehat{\nu}})$ is given by

$$B_R(E_{\mathbb{R}},\|\cdot\|_{\widehat{\nu}})=\mathrm{span}_{\mathbb{R}}(\{y\in E\,:\,\|y\|\leqslant R\}).$$

By definition, for any $i\in\{1,\ldots,\mathrm{rk}_{\mathbb{Z}}(E)\}$, one has

$$\lambda_i(E_{\mathbb{R}},\|\cdot\|_{\widehat{\nu}})=\widehat{\nu}_i(E,\|\cdot\|). \tag{2.3}$$

This construction thus relates the successive minima of $(E,\|\cdot\|)$ to an ultrametric norm on $E_{\mathbb{R}}$. A similar construction exists for successive slopes. To explain this point, the following proposition is important. In order to simplify the presentation, for any non-zero Euclidean lattice $\overline{F}$, we denote by $\widehat{\nu}_{\min}(\overline{F})$ the last minimum of $\overline{F}$, namely

$$\widehat{\nu}_{\min}(\overline{F}):=\widehat{\nu}_{\mathrm{rk}_{\mathbb{Z}}(F)}(\overline{F}).$$

Proposition 2.4. — *Let $(E,\|\cdot\|)$ be a Euclidean lattice. For any $R>0$, the central ball of radius R of $(E,\|\cdot\|_{\widehat{\nu}})$ is given by*

$$\sum_{\substack{\{0\}\neq F\subset E\\ \widehat{\nu}_{\min}(\overline{F})\geqslant-\ln(R)}}F_{\mathbb{R}},$$

where F runs over the set of all non-zero subgroups of E such that $\widehat{\nu}_{\min}(\overline{F})\geqslant -\ln(R)$.

Proof. — Let F be a non-zero subgroup of E such that $\widehat{\nu}_{\min}(\overline{F})\geqslant-\ln(R)$, then F admits a basis over $\mathbb{Z}$ which consists of elements of norm (with respect to $\|\cdot\|$) $\leqslant R$. Therefore, any vector in $F_{\mathbb{R}}$ is generated over $\mathbb{R}$ by elements $x\in F$ such that $\|x\|\leqslant R$, which shows that $F_{\mathbb{R}}\subset B_R(E_{\mathbb{R}},\|\cdot\|_{\widehat{\nu}})$. Conversely, as $B_R(E_{\mathbb{R}},\|\cdot\|_{\widehat{\nu}})$ is spanned over $\mathbb{R}$ by vectors $s\in E$ such that $\|s\|\leqslant R$, we obtain that

$$\widehat{\nu}_{\min}(\overline{B_R(E_{\mathbb{R}},\|\cdot\|_{\widehat{\nu}})\cap E})\geqslant-\ln(R),$$

where the last minimum is defined to be $+\infty$ by convention when the intersection $B_R(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\nu}}) \cap E$ is $\{0\}$. Since $B_R(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\nu}})$ is spanned over $\mathbb{R}$ by $B_R(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\nu}}) \cap E$, we obtain that

$$B_R(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\nu}}) \subset \sum_{\substack{\{0\}\neq F\subset E\\ \widehat{\nu}_{\min}(\overline{F})\geqslant -\ln(R)}} F_{\mathbb{R}}.$$

□

Definition 2.5. — For any non-zero Euclidean lattice $\overline{F}$, we denote by $\widehat{\mu}_{\min}(\overline{F})$ the *minimal slope* of $\overline{F}$, which is defined as

$$\widehat{\mu}_{\min}(\overline{F}) := \widehat{\mu}_{\mathrm{rk}_{\mathbb{Z}}(F)}(\overline{F}).$$

If $\overline{E}$ is a Euclidean lattice, we denote by $\|\cdot\|_{\widehat{\mu}}$ the norm on $E_{\mathbb{R}}$ (where $\mathbb{R}$ is equipped with the trivial absolute value) such that the ball of radius $R > 0$ is given by

$$\sum_{\substack{\{0\}\neq F\subset E\\ \widehat{\mu}_{\min}(\overline{F})\geqslant -\ln(R)}} F_{\mathbb{R}}.$$

The following proposition is fundamental in the reformulation of Harder-Narasimhan filtration in terms of geometry over a trivially valued field. We refer the readers to [**7**, §2.2.2] for a proof in the setting of $\mathbb{R}$-filtrations, see also [**11**, Remark 1.1.40].

Proposition 2.6. — *Let $(E, \|\cdot\|)$ be a non-zero Euclidean lattice.*

1. *For any $i \in \{1, \ldots, \mathrm{rk}_{\mathbb{Z}}(E)\}$, one has $\lambda_i(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\mu}}) = \widehat{\mu}_i(E, \|\cdot\|)$.*
2. *If $0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_d = E$ is the Harder-Narasimhan filtration of $(E, \|\cdot\|)$, then*

$$0 = E_{0,\mathbb{R}} \subsetneq E_{1,\mathbb{R}} \subsetneq \ldots \subsetneq E_{d,\mathbb{R}} = E_{\mathbb{R}}$$

is the Harder-Narasimhan filtration of $(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\mu}})$.

Definition 2.7. — Let n be a positive integer. We denote by $\delta(n)$ the following value

$$\sup\{\widehat{\mu}_{\min}(\overline{E}) - \widehat{\nu}_{\min}(\overline{E}) \,:\, \overline{E} \text{ is a Euclidean lattice of rank } n\}.$$

Note that the function $n \mapsto \delta(n)$ is increasing. In fact, if $\overline{E}$ and $\overline{F}$ are two Euclidean lattices, one has

$$\widehat{\mu}_{\min}(\overline{E} \oplus \overline{F}) = \min(\widehat{\mu}_{\min}(\overline{E}), \widehat{\mu}_{\min}(F)), \tag{2.4}$$

$$\widehat{\nu}_{\min}(\overline{E} \oplus \overline{F}) \leqslant \min(\widehat{\nu}_{\min}(\overline{E}), \widehat{\nu}_{\min}(\overline{F})), \tag{2.5}$$

where (2.4) comes from the interpretation of $\widehat{\mu}_{\min}$ as the infimum of slopes of quotient lattices.

We can now explain the proof of the comparison principle between minima and slopes.

Theorem 2.8. — *Let $\overline{E}$ be a non-zero Euclidean lattice and n be the rank of E over $\mathbb{Z}$. For any $i \in \{1, \ldots, n\}$ one has*

$$0 \leqslant \widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i(\overline{E}) \leqslant \delta(n). \tag{2.6}$$

Proof. — For any non-zero subgroup F of E, one has

$$0 \leqslant \widehat{\mu}_{\min}(\overline{F}) - \widehat{\nu}_{\min}(\overline{F}) \leqslant \delta(\mathrm{rk}_{\mathbb{Z}}(F)) \leqslant \delta(n),$$

where the first inequality is a consequence of Hadamard's inequality, see [**2**] and [**10**, Proposition 3.4]. By Proposition 2.4 and the construction of $\|\cdot\|_{\widehat{\nu}}$, one has

$$\|\cdot\|_{\widehat{\mu}} \leqslant \|\cdot\|_{\widehat{\nu}} \leqslant \mathrm{e}^{\delta(n)}\|\cdot\|_{\widehat{\mu}}.$$

By Proposition 2.3, for $i \in \{1, \ldots, n\}$, one has

$$\lambda_i(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\nu}}) \leqslant \lambda_i(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\mu}}) \leqslant \lambda_i(E_{\mathbb{R}}, \|\cdot\|_{\widehat{\nu}}) + \delta(n).$$

By Proposition 2.6 and the formula (2.3), we obtain (2.6). □

3. Transference problem

Theorem 2.8 reduces the comparison problem between successive minima and slopes to an upper bound of the difference between the minimal slope and the last minimum. If we consider the difference between the maximal slope and the first minimum instead, Minkowski's first theorem implies that, for any non-zero Euclidean lattice $\overline{E}$ of rank n, one has

$$\widehat{\mu}_1(\overline{E}) - \widehat{\nu}_1(\overline{E}) \leqslant \ln(2) - \frac{1}{n}\ln(v_n),$$

where v_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. One might expect that the upper bound of $\widehat{\mu}_{\min}(\overline{E}) - \widehat{\nu}_{\min}(\overline{E})$ would follow from a classic result of geometry of numbers. However, this problem seems to be much deeper and get involved more tools. In this section, we give a simple proof of Banaszczyk's transference theorem and deduce an upper bound for the function $\delta(\cdot)$. In the rest of the section, we fix a non-zero Euclidean lattice $\overline{E} = (E, \|\cdot\|)$. Let n be the rank of E over $\mathbb{Z}$ and $\langle\ ,\ \rangle$ be the Euclidean inner product on $E_{\mathbb{R}}$ associated with the norm $\|\cdot\|$. Let $E^{\vee}$ be the dual $\mathbb{Z}$-module of E. We equip $(E^{\vee})_{\mathbb{R}} \cong (E_{\mathbb{R}})^{\vee}$ with the dual Euclidean norm $\|\cdot\|_*$. Note that $(E^{\vee}, \|\cdot\|_*)$ is a Euclidean lattice, and the following equality holds (see for example [**11**, Proposition 4.3.8] for a proof):

$$\widehat{\deg}(E, \|\cdot\|) + \widehat{\deg}(E^{\vee}, \|\cdot\|_*) = 0. \tag{3.1}$$

Definition 3.1. — For any subset A of $E_{\mathbb{R}}$, we denote by $\rho(A)$ the sum

$$\sum_{x\in A} \mathrm{e}^{-\pi\|x\|^2} \in [0,+\infty].$$

Similarly, if B is a subset of $E_{\mathbb{R}}^{\vee}$, we denote by $\rho(B)$ the sum

$$\sum_{\alpha\in B} \mathrm{e}^{-\pi\|\alpha\|_*^2} \in [0,+\infty].$$

Proposition 3.2. — *For any $x \in E_{\mathbb{R}}$ one has $\rho(E+x) < +\infty$, where by definition*

$$E + x := \{u + x \,:\, u\in E\}.$$

Moreover, the function $\rho_E : E_{\mathbb{R}} \to \mathbb{R}$ defined as $\rho_E(x) := \rho(E+x)$ is smooth and E-periodic.

Proof. — Let $(e_j)_{j=1}^n$ be a basis of E. As all norms on $\mathbb{R}^n$ are equivalent, there exists $c > 0$ such that

$$\forall\,(a_1,\ldots,a_n)\in\mathbb{R}^n,\quad \|a_1e_1+\cdots+a_ne_n\|^2 \geqslant c(a_1^2+\cdots+a_n^2).$$

Therefore, for $x = b_1e_1+\cdots+b_ne_n \in E_{\mathbb{R}}$,

$$\begin{aligned}\rho(E+x) &\leqslant \sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} \mathrm{e}^{-c\pi((a_1+b_1)^2+\cdots+(a_n+b_n)^2)}\\ &= \prod_{\ell=1}^n \Big(\sum_{a\in\mathbb{Z}} \mathrm{e}^{-c\pi(a+b_\ell)^2}\Big) < +\infty.\end{aligned}$$

The E-periodicity of the function $\rho_E(\cdot)$ follows directly from the definition. Finally, any formal partial derivative (possibly of higher order) of the series defining ρ_E can be written as

$$\sum_{u\in E} P(u+x)\mathrm{e}^{\pi\|u+x\|^2} \tag{3.2}$$

where $P(\cdot)$ is a polynomial on $E_{\mathbb{R}}$. Note that for any $\delta > 0$ there exists $C > 0$ such that $\|P(y)\| \leqslant C\mathrm{e}^{\delta\|y\|^2}$ for any $y \in \mathbb{R}^n$. Therefore the series (3.2) converges uniformly on $\mathbb{R}^n$. Hence ρ_E is smooth on $E_{\mathbb{R}}$. □

Proposition 3.3. — *For any $y \in E_{\mathbb{R}}$, one has*

$$\int_{E_{\mathbb{R}}} \mathrm{e}^{-\pi\|x\|^2-2\pi i\langle x,y\rangle}\,\mathrm{d}x = \mathrm{e}^{-\pi\|y\|^2}, \tag{3.3}$$

where $\mathrm{d}x$ denotes the unique Haar measure on $E_{\mathbb{R}}$ such that the parallelotope spanned by an orthonormal basis of $E_{\mathbb{R}}$ has volume 1.

Proof. — Recall that for any $\theta \in \mathbb{R}$ one has

$$\int_{\mathbb{R}} \mathrm{e}^{-\pi t^2 - 2\pi i\theta t}\,\mathrm{d}t = \mathrm{e}^{-\pi\theta^2}. \tag{3.4}$$

This can for example be deduced from the classic equality $\int_{\mathbb{R}} \mathrm{e}^{-\pi t^2}\,\mathrm{d}t = 1$ via the asymptotic study (when $R \in \mathbb{R}_{>0}$, $R \to +\infty$) on the integral of the analytic function $z \mapsto \mathrm{e}^{-\pi z^2}$ along the contours formed by the segments linking $-R$, R, $R+i\theta$, $-R+i\theta$, $-R$ successively. Moreover, the formula (3.3) in the particular case where y vanishes follows also from the equality $\int_{\mathbb{R}} \mathrm{e}^{-\pi t^2}\,\mathrm{d}t = 1$ by Fubini's theorem.

In the following, we assume that y is non-zero. We pick an orthonormal basis of $E_{\mathbb{R}}$ which contains the vector $\|y\|^{-1}y$. By a change of variables, we obtain, using Fubini's theorem, that

$$\int_{E_{\mathbb{R}}} \mathrm{e}^{-\pi\|x\|^2 - 2\pi i\langle x,y\rangle}\,\mathrm{d}x = \bigg(\int_{\mathbb{R}} \mathrm{e}^{-\pi t^2}\,\mathrm{d}t\bigg)^{n-1} \int_{\mathbb{R}} \mathrm{e}^{-\pi t^2 - 2\pi i t\|y\|}\mathrm{d}t = \mathrm{e}^{-\pi\|y\|^2},$$

where the last equality comes from (3.4). The proposition is thus proved. □

The following proposition gives the Fourier expansion of the function $\rho_E(\cdot)$.

Proposition 3.4. — *For any $x \in E_{\mathbb{R}}$, one has*

$$\rho(E+x) = \exp\big(\widehat{\deg}(\overline{E})\big) \sum_{\alpha \in E^{\vee}} \mathrm{e}^{2\pi i\alpha(x) - \pi\|\alpha\|_*^2}, \tag{3.5}$$

where in the expression $\alpha(x)$, we consider α as a linear form on $E_{\mathbb{R}}$ by the identification $(E^{\vee})_{\mathbb{R}} \cong (E_{\mathbb{R}})^{\vee}$. In particular the following equality holds

$$\rho(E) = \exp\big(\widehat{\deg}(\overline{E})\big)\rho(E^{\vee}). \tag{3.6}$$

Proof. — Let D be a fundamental domain of the lattice E and $L^2(D,\mathbb{C})$ be the space of square-integrable complex-valued function on D. We equipped it with the following inner product

$$\forall\,(f,g) \in L^2(D,\mathbb{C}), \quad \langle f,g\rangle_{L^2} := \exp\big(\widehat{\deg}(\overline{E})\big) \int_D f(x)\overline{g(x)}\,\mathrm{d}x.$$

Then $(\mathrm{e}^{2\pi i\alpha(\cdot)})_{\alpha\in E^{\vee}}$ forms an orthonormal basis of $(L^2(D,\mathbb{C}), \langle\ ,\ \rangle_{L^2})$. Therefore the series of functions

$$\exp\big(\widehat{\deg}(\overline{E})\big) \sum_{\alpha\in E^{\vee}} \mathrm{e}^{2\pi i\alpha(\cdot)} \int_D \rho(E+y)\mathrm{e}^{-2\pi i\alpha(y)}\,\mathrm{d}y \tag{3.7}$$

converges in $L^2(D,\mathbb{C})$ to the restriction of ρ_E on D. Moreover, one has

$$\int_D \rho(E+y)\mathrm{e}^{-2\pi i\alpha(y)}\,\mathrm{d}y = \int_D \sum_{u\in E} \mathrm{e}^{-\pi\|y+u\|^2-2\pi i\alpha(y)}\,\mathrm{d}y$$
$$= \int_{E_{\mathbb{R}}} \mathrm{e}^{-\pi\|x\|^2-2\pi i\alpha(x)}\,\mathrm{d}x = \mathrm{e}^{-\pi\|\alpha\|_*^2},$$

where the last equality comes from (3.3). Since ρ_E is continuous (see Proposition 3.2) and $\sum_{\alpha\in E^\vee}\mathrm{e}^{-\pi\|\alpha\|_*^2}<+\infty$, the series (3.7) actually converges uniformly to $\rho_E|_D$. The equality (3.5) is thus proved.

The particular case of (3.5) with $x=0$ gives (3.6). □

Corollary 3.5. — (a) *For any $x\in E_{\mathbb{R}}$ one has $\rho(E+x)\leqslant\rho(E)$.*
(b) *For any $x\in E_{\mathbb{R}}$ and any $t\in\,]0,1[$ one has $\rho(t(E+x))\leqslant t^{-n}\rho(E)$, where n is the rank of E over $\mathbb{Z}$.*

Proof. — By (3.5), one has

$$\rho(E+x)\leqslant\exp\big(\widehat{\deg}(\overline{E})\big)\sum_{\alpha\in E^\vee}\big|\mathrm{e}^{2\pi i\alpha(x)-\pi\|\alpha\|_*^2}\big|$$
$$=\exp\big(\widehat{\deg}(\overline{E})\big)\sum_{\alpha\in E^\vee}\mathrm{e}^{-\pi\|\alpha\|_*^2}=\rho(E),$$

which proves (a), where the last equality comes from (3.5).

For any $t>0$, the subgroup tE of $E_{\mathbb{R}}$ forms a Euclidean lattice in $E_{\mathbb{R}}$. The subgroup $t^{-1}E^\vee$ of $E_{\mathbb{R}}^\vee$ forms also a Euclidean lattice in $E_{\mathbb{R}}^\vee$, which identifies with the dual lattice of tE. Therefore, for any $x\in\mathbb{R}^n$ and any $t\in\,]0,1[$ one has

$$\rho(t(E+x))\leqslant\rho(tE)=\exp\big(\widehat{\deg}(\overline{tE})\big)\rho(t^{-1}E^\vee)$$
$$\leqslant\exp\big(\widehat{\deg}(\overline{E})-nt\big)\rho(E^\vee)=\mathrm{e}^{-nt}\rho(E),$$

where the first inequality comes from (a) and the second one comes from the hypothesis $t\in\,]0,1[$, and the equalities comes from (3.5). The assertion (b) is thus proved. □

The following lemma is a key argument in the proof of Banaszczyk's transference theorem. For any $r>0$, we denote by B_r the central ball of radius r in $(E_{\mathbb{R}},\|\cdot\|)$. Namely $B_r:=\{y\in E_{\mathbb{R}}\,:\,\|y\|\leqslant r\}$.

Lemma 3.6. — *For all $r>0$, $t\in\,]0,1[$ and $x\in E_{\mathbb{R}}$, one has*

$$\rho((E+x)\setmin B_r)\leqslant\mathrm{e}^{-\pi(1-t)r^2}t^{-\frac{n}{2}}\rho(E),\tag{3.8}$$

where $n=\mathrm{rk}_{\mathbb{Z}}(E)$. If $r\geqslant\sqrt{\frac{n}{2\pi}}$, then one has

$$\rho((E+x)\setminus B_r)\leqslant\mathrm{e}^{\frac{n}{2}-\pi r^2}\Big(\frac{n}{2\pi r^2}\Big)^{-\frac{n}{2}}\rho(E).\tag{3.9}$$

Proof. — One has

$$\rho((E+x)\setminus B_r)=\sum_{\substack{u\in E\\ \|u+x\|\geqslant r}}\mathrm{e}^{-\pi\|u+x\|^2}=\sum_{\substack{u\in E\\ \|u+x\|\geqslant r}}\mathrm{e}^{-\pi t\|u+x\|^2}\mathrm{e}^{-\pi(1-t)\|u+x\|^2}$$
$$\leqslant \mathrm{e}^{-\pi(1-t)r^2}\rho(t^{\frac{1}{2}}(E+x))\leqslant \mathrm{e}^{-\pi(1-t)r^2}t^{-\frac{n}{2}}\rho(E),$$

where the last inequality comes from Corollary 3.5 (b). Thus the inequality (3.8) is proved. Taking $t=n/(2\pi r^2)$ we obtain (3.9). □

***Theorem 3.7* (Banaszczyk).** — *Let $(E,\|\cdot\|)$ be a non-zero Euclidean lattice and $n=\mathrm{rk}_{\mathbb{Z}}(E)$. One has*

$$\widehat{\nu}_n(E,\|\cdot\|)+\widehat{\nu}_1(E^\vee,\|\cdot\|_*)\geqslant-\ln(n). \tag{3.10}$$

Proof. — In the case where $n=1$, one has

$$\widehat{\nu}_1(E,\|\cdot\|)=\widehat{\deg}(E,\|\cdot\|)\quad\text{and}\quad\widehat{\nu}_1(E^\vee,\|\cdot\|_*)=\widehat{\deg}(E^\vee,\|\cdot\|_*).$$

Hence (3.10) follows from (3.1).

In the rest of the proof we assume that $n\geqslant 2$. Denote by r the unique positive number in $[\sqrt{\frac{n}{2\pi}},+\infty[$ such that [1]

$$\exp\Big(\frac{n}{2}-\pi r^2\Big)\Big(\frac{n}{2\pi r^2}\Big)^{-\frac{n}{2}}=\frac{1}{4}. \tag{3.11}$$

Let t_0 be the unique solution on $[0,+\infty[$ of the equation

$$t\mathrm{e}^{1-t}=\frac{1}{4}.$$

A numerical computation shows that $t_0<3.7$. Since $n\geqslant 2$, we obtain that $(t_0\mathrm{e}^{1-t_0})^{\frac{n}{2}}\leqslant\frac{1}{4}$ and hence $2\pi r^2/n<t_0$ and $r<\sqrt{nt_0/2\pi}$.

For any $b>0$ one has

$$\widehat{\nu}_1(b^{-1}E^\vee,\|\cdot\|_*)=\widehat{\nu}_1(E^\vee,\|\cdot\|_*)-\ln(b)$$

and

$$\widehat{\nu}_n(bE,\|\cdot\|)=\widehat{\nu}_n(E,\|\cdot\|)+\ln(b).$$

Therefore by dilating the lattice E we may assume without loss of generality that $\widehat{\nu}_1(E^\vee,\|\cdot\|_*)=-\ln(r)$. We reason by contradiction and assume that

$$\widehat{\nu}_n(E,\|\cdot\|)<\ln(r)-\ln(n). \tag{3.12}$$

1. Note that the function $t\mapsto(\mathrm{e}^{1-t}t)^{\frac{n}{2}}$ is strictly decreasing on $[1,+\infty[$, takes value 1 at $t=1$, and tends to 0 when $t\to+\infty$.

By (3.5), for any $x \in E_{\mathbb{R}}$ one has

$$\begin{aligned}\rho(E+x) &= \exp\big(\widehat{\deg}(\overline{E})\big)\sum_{\alpha\in E^\vee} \mathrm{e}^{2\pi i\alpha(x)-\pi\|\alpha\|_*^2}\\ &\geqslant \exp\big(\widehat{\deg}(\overline{E})\big)(1-\rho(E^\vee\setminus B_r^*))\\ &\geqslant \exp\big(\widehat{\deg}(\overline{E})\big)\Big(1-\frac{1}{4}\rho(E^\vee)\Big),\end{aligned}\tag{3.13}$$

where $B_r^* := \{\varphi\in E_{\mathbb{R}}^\vee \,:\, \|\varphi\|_*\leqslant r\}$ denotes the central ball of radius r in the dual Euclidean space $(E_{\mathbb{R}}^\vee, \|\cdot\|_*)$, and the last inequality comes from (3.9) and (3.11), which lead to $\rho(E^\vee\setminus B_r^*)\leqslant \frac{1}{4}\rho(E^\vee)$. As

$$\rho(E^\vee) = 1+\rho(E^\vee\setminus B_r^*)\leqslant 1+\frac{1}{4}\rho(E^\vee),$$

one has $\rho(E^\vee)\leqslant\frac{4}{3}$. Hence we deduce from (3.13) that

$$\rho(E+x)\geqslant \frac{2}{3}\cdot\exp\big(\widehat{\deg}(\overline{E})\big) = \frac{2}{3}\cdot\frac{\rho(E)}{\rho(E^\vee)}\geqslant\frac{1}{2}\rho(E).\tag{3.14}$$

Let $s := \exp(-\widehat{\nu}_n(E,\|\cdot\|))$ and H be the vector subspace of $E_{\mathbb{R}}$ generated by $B_s\cap E$. By definition, H is a strict vector subspace of $E_{\mathbb{R}}$. We pick an element $x\in H^\perp$ such that $\|x\| = 2/3$. Then one has $\mathrm{e}^{-\pi\|x\|^2}<\frac{1}{4}$ and

$$\rho((E\cap H)+x) = \mathrm{e}^{-\pi\|x\|^2}\rho(E\cap H)\leqslant \mathrm{e}^{-\pi\|x\|^2}\rho(E)<\frac{1}{4}\rho(E).\tag{3.15}$$

Note that

$$s-\|x\|>\frac{n}{r}-\frac{2}{3} = r+\Big(\frac{n}{r}-r-\frac{2}{3}\Big).$$

Note that, since $r<\sqrt{nt_0/2\pi}$ and $n\geqslant 2$, one has

$$\frac{n}{r}-r-\frac{2}{3}>\sqrt{\frac{2\pi n}{t_0}}-\sqrt{\frac{nt_0}{2\pi}}-\frac{2}{3}\geqslant\sqrt{\frac{4\pi}{t_0}}-\sqrt{\frac{t_0}{\pi}}-\frac{2}{3}>0$$

where the last inequality comes from the fact that $t_0<3.7$. Therefore, by (3.9) one has

$$\rho((E\setminus H)+x)\leqslant\rho((E+x)\setminus B_{s-\|x\|})\leqslant\rho((E+x)\setminus B_r)\leqslant\frac{1}{4}\rho(E).\tag{3.16}$$

The inequalities (3.15) and (3.16) lead to $\rho(E+x)<\frac{1}{2}\rho(E)$, which contradicts (3.14). The theorem is thus proved. □

Corollary 3.8. — *For any non-zero Euclidean lattice $\overline{E}$ one has*

$$\widehat{\mu}_{\min}(E,\|\cdot\|)-\widehat{\nu}_{\min}(E,\|\cdot\|)\leqslant\ln(\mathrm{rk}_{\mathbb{Z}}(E)).$$

In other words, for any $n\in\mathbb{N}_{\geqslant 1}$, $\delta(n)\leqslant\ln(n)$.

Proof. — Let n be the rank of E over $\mathbb{Z}$. By (2.6), one has

$$\widehat{\mu}_1(E^\vee, \|\cdot\|_*) \geqslant \widehat{\nu}_1(E^\vee, \|\cdot\|_*),$$

which implies, by (3.1) and (3.10), that

$$-\widehat{\mu}_n(E, \|\cdot\|) \geqslant -\widehat{\nu}_n(E, \|\cdot\|) - \ln(n).$$

□

4. Further discussions

4.1. Adelic vector bundles. — The comparison principle (Proposition 2.3) actually applies to general number fields and allows to compare several arithmetic invariants.

Let K be a number field and let M_K be the set of all places of K. For $v \in M_K$, we denote by $|\cdot|_v$ the absolute value in the class v which extends either the usual absolute value or a certain p-adic absolute on $\mathbb{Q}$ (in the former case, v is said to be *Archimedean*, while in the latter case, v is said to be *non-Archimedean*). We denote by K_v and $\mathbb{Q}_v$ the completion of K and $\mathbb{Q}$ with respect to the absolute value $|\cdot|_v$, respectively. Note that the product formula shows that

$$\forall\, a \in K \setminus \{0\}, \quad \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \ln |a|_v = 0, \tag{4.1}$$

As *adelic vector bundle* on $\operatorname{Spec} K$ (cf. [**14**, §3]), we refer to the datum $\overline{E}$ consisting of a finite-dimensional vector space E over K, equipped with a family $(\|\cdot\|_v)_{v \in M_K}$ of norms, indexed by the places of K, which satisfies the following conditions:

(1) for any $v \in M_K$, $\|\cdot\|_v$ is a norm on $E_{K_v} = E \otimes_K K_v$,
(2) there exists a basis $(e_i)_{i=1}^n$ of E such that the relation

$$\forall\, (\lambda_1, \ldots, \lambda_n) \in K_v^n, \quad \|\lambda_1 e_1 + \cdots + \lambda_n e_n\| = \max_{i \in \{1,\ldots,n\}} |\lambda_i| \cdot \|e_i\|$$

holds for all but finitely many places v.

If $\|\cdot\|_v$ is induced by an inner product when the place v is Archimedean, and is ultrametric when v is non-Archimedean, we say that the adelic vector bundle $\overline{E}$ is *Hermitian*. If the norm $\|\cdot\|_v$ is pure (namely its image is contained in that of $|\cdot|_v$) for any non-Archimedean place v, we say that the adelic vector bundle $\overline{E}$ is *pure*. In the case where E is a one-dimensional K-vector space, we say that $\overline{E}$ is an *adelic line bundle* on $\operatorname{Spec} K$. Note that an adelic line bundle is necessarily Hermitian.

Let $\overline{E}$ be a Hermitian adelic vector bundle on $\operatorname{Spec} K$ and K'/K be a finite extension. We denote by $\overline{E} \otimes_K K'$ the Hermitian adelic vector bundle on

$\operatorname{Spec} K'$ consisting of the K'-vector space $E \otimes_K K'$ and norms $(\|\cdot\|_w)_{w \in M_{K'}}$, such that, for any $v \in M_K$ and any $w \in M_{K'}$ lying over v,

(a) in the case where v is non-Archimedean, $\|\cdot\|_w$ is the largest ultrametric norm on $E'_{K'_w} \cong E_{K_v} \otimes_{K_v} K'_w$ extending $\|\cdot\|_{v,**}$,

(b) in the case where v is Archimedean, $\|\cdot\|_w$ is the Hilbert-Schmidt tensor product of $\|\cdot\|_v$ and the absolute value $|\cdot|_w$ on K'_w.

We refer the readers to [**11**, §1.3] for more details about the extension of norms.

Let $\overline{E} = (E, (\|\cdot\|_v)_{v \in M_K})$ be a Hermitian adelic vector bundle on $\operatorname{Spec} K$. For any non-zero element s of E, we define

$$\widehat{\deg}_{\overline{E}}(s) := -\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \ln\|s\|_v.$$

Note that the product formula (4.1) shows that

$$\forall\, a \in K \setminus \{0\}, \quad \widehat{\deg}_{\overline{E}}(as) = \widehat{\deg}_{\overline{E}}(s).$$

If K' is a finite extension of K and if s is a non-zero element of $E_{K'}$, we use the simplified notation $\widehat{\deg}_{\overline{E}}(s)$ to denote $\widehat{\deg}_{\overline{E} \otimes_K K'}(s)$. Since the sum of local degrees of a finite separable extension is equal to the global degree (see for example [**23**] Chapter II, Corollary 8.4), if $K''/K'/K$ are successive finite extensions and if s is a non-zero element of $E_{K'}$, then one has

$$\widehat{\deg}_{\overline{E} \otimes_K K'}(s) = \widehat{\deg}_{\overline{E} \otimes_K K''}(s),$$

where on the right-hand side of the formula, we consider s as a non-zero element of $E \otimes_K K''$. This observation allows to consider $\widehat{\deg}_{\overline{E}}(\cdot)$ as a function on $E \otimes_K K^{\mathrm{a}}$, where K^{a} is a fixed algebraic closure of K.

Given an adelic vector bundle $\overline{E} = (E, (\|\cdot\|_v)_{v \in M_K})$ on $\operatorname{Spec} K$, we construct an adelic line bundle $\det(\overline{E})$ as follows. The vector space part of $\det(\overline{E})$ is give by the maximal exterior power of E. For any $v \in M_K$, we equip $\det(E_{K_v}) \cong \det(E) \otimes_K K_v$ with the determinant norm $\|\cdot\|_{v,\det}$, defined as

$$\forall\, \eta \in E_{K_v}, \quad \|\eta\|_{v,\det} = \inf_{\substack{(x_i)_{i=1}^n \in E_{K_v}^n \\ \eta = x_1 \wedge \cdots \wedge x_n}} \|x_1\|_v \cdots \|x_n\|_v.$$

The *Arakelov degree* of $\overline{E}$ is defined as

$$\widehat{\deg}(\overline{E}) := -\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \ln\|\eta\|_{v,\det},$$

where η is a non-zero element of $\det(E)$. Note that this definition does not depend on the choice of the non-zero determinant vector η (this is a consequence of the product formula (4.1)). Note that the Arakelov degree function thus constructed satisfies the inequalities in the form of (1.2) and

(1.3) for Hermitian adelic vector bundles. Hence an analogue of Harder-Narasimhan's theorem is valid and leads to the notion of successive slopes $\widehat{\mu}_i(\overline{E})$ for a Hermitian adelic vector bundle $\overline{E}$.

The notion of minima can also be naturally generalised to the setting adelic vector bundles. There actually exist several versions of successive minima. We recall below some of them. We fix a non-zero adelic vector bundle $\overline{E}$ and let n be the rank of E over K and i be an arbitrary element of $\{1,\ldots,n\}$.

(1) The *Bombieri-Vaaler* i^{th} *minimum* of $\overline{E}$ is defined as the supremum $\widehat{\nu}_i^{\mathrm{BV}}(\overline{E})$ of the set of $t\in\mathbb{R}$ such that there exists a family of K-linearly independent vectors $\{s_1,\ldots,s_i\}$ in E which satisfies the following condition: for any $j\in\{1,\ldots,i\}$, $\|s_j\|_v\leqslant 1$ for any non-Archimedean place v and $\|s_j\|_\sigma\leqslant \mathrm{e}^{-t}$ for any Archimedean places σ.

(2) The *Roy-Thunder* i^{th} *minimum* of $\overline{E}$ is defined as the supremum $\widehat{\nu}_i^{\mathrm{RT}}(\overline{E})$ of the set of $t\in\mathbb{R}$ such that there exists a K-linearly independent family $\{s_1,\ldots,s_i\}$ of vectors in E satisfying $\widehat{\deg}_{\overline{E}}(s_j)\geqslant t$ for any $j\in\{1,\ldots,i\}$.

(3) The *absolute* i^{th} *minimum* of $\overline{E}$ is defined as the supremum $\widehat{\nu}_i^{\mathrm{abs}}(\overline{E})$ of the set of $t\in\mathbb{R}$ such that there exists a K^{a}-linearly independent family $\{s_1,\ldots,s_i\}$ of vectors in $E\otimes_K K^{\mathrm{a}}$ satisfying $\widehat{\deg}_{\overline{E}}(s_j)\geqslant t$ for any $j\in\{1,\ldots,i\}$.

Note that, if s is a non-zero element of E such that $\|s\|\leqslant 1$ for any non-Archimedean place v and $\|s\|_\sigma\leqslant\mathrm{e}^{-t}$ for any Archimedean place σ, then one has

$$\widehat{\deg}_{\overline{E}}(s)\geqslant -\sum_{\sigma\in M_{K,\infty}}\frac{[K_\sigma:\mathbb{Q}_\sigma]}{[K:\mathbb{Q}]}\ln\|s\|_\sigma\geqslant t,$$

where $M_{K,\infty}$ denotes the set of Archimedean places of K. Therefore, for any $i\in\{1,\ldots,n\}$, the following inequalities hold

$$\widehat{\nu}_i^{\mathrm{BV}}(\overline{E})\leqslant\widehat{\nu}_i^{\mathrm{RT}}(\overline{E})\leqslant\widehat{\nu}_i^{\mathrm{abs}}(\overline{E}).$$

Moreover, if K' is a finite extension of K, and $\{s_1,\ldots,s_n\}$ is a basis of $E\otimes_K K'$, then Hadamard's inequality (see for example [**11**, Proposition 1.1.66] for a proof) shows that

$$\begin{aligned}\widehat{\deg}(\overline{E})=\widehat{\deg}(\overline{E}\otimes_K K')&=-\sum_{v\in M_{K'}}\frac{[K'_v:\mathbb{Q}_v]}{[K':\mathbb{Q}]}\ln\|s_1\wedge\cdots\wedge s_n\|_v\\&\geqslant-\sum_{i=1}^n\sum_{v\in M_{K'}}\frac{[K'_v:\mathbb{Q}_v]}{[K':\mathbb{Q}]}\ln\|s_i\|_v=\sum_{i=1}^n\widehat{\deg}_{\overline{E}}(s_i),\end{aligned}$$

which shows that $\widehat{\mu}(\overline{E})\geqslant\widehat{\nu}_{\min}^{\mathrm{abs}}(\overline{E})$. Applying this inequality to quotient adelic vector bundles of $\overline{E}$ leads to

$$\widehat{\mu}_{\min}(\overline{E})\geqslant\widehat{\nu}_{\min}^{\mathrm{abs}}(\overline{E}).$$

Note that the analogue of Propositions 2.4 and 2.6 (for any of the above minima and for the Harder-Narasimhan filtration respectively) in the setting of Hermitian adelic vector bundle is still true. We obtain that, for any non-zero *pure* Hermitian adelic vector bundle $\overline{E}$ and any $i \in \{1, \ldots, \mathrm{rk}_{\mathbb{Z}}(n)\}$, one has

$$0 \leqslant \widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i^{\mathrm{BV}}(\overline{E}) \leqslant \delta^{\mathrm{BV}}(n),$$
$$0 \leqslant \widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i^{\mathrm{RT}}(\overline{E}) \leqslant \delta^{\mathrm{RT}}(n),$$

where

$$\delta^{\mathrm{BV}}(n) := \sup_{\overline{F}}(\widehat{\mu}_{\min}(\overline{F}) - \widehat{\nu}_{\min}^{\mathrm{BV}}(\overline{F})),$$
$$\delta^{\mathrm{RT}}(n) := \sup_{\overline{F}}(\widehat{\mu}_{\min}(\overline{F}) - \widehat{\nu}_{\min}^{\mathrm{RT}}(\overline{F})),$$

with $\overline{F}$ running over the set of *pure* Hermitian adelic vector bundles of rank n over $\mathrm{Spec}\, K$. Similarly, for any (non-necessarily pure) non-zero Hermitian adelic vector bundle $\overline{E}$ and any $i \in \{1, \ldots, \mathrm{rk}_{\mathbb{Z}}(E)\}$, one has

$$0 \leqslant \widehat{\mu}_i(\overline{E}) - \widehat{\nu}_i^{\mathrm{abs}}(\overline{E}) \leqslant \delta^{\mathrm{abs}}(n),$$

where

$$\delta^{\mathrm{abs}}(n) := \sup_{\overline{F}}(\widehat{\mu}_{\min}(\overline{F}) - \widehat{\nu}_{\min}^{\mathrm{abs}}(\overline{F})),$$

with $\overline{F}$ running over the set of Hermitian adelic vector bundles of rank n over the spectrum of a number field.

Given a Hermitian adelic vector bundle $\overline{E}$ over $\mathrm{Spec}\, K$, we construct a Euclidean lattice as follows. We consider the real vector space

$$\bigoplus_{\sigma \in M_{K,\infty}} E \otimes_K K_\sigma,$$

which is of dimension $\dim_K(E)[K : \mathbb{Q}]$. We equip this vector space with the orthogonal direct sum of the norms $\|\cdot\|_\sigma$. Then

$$\mathcal{E} := \{s \in E \,:\, \forall\, v \in M_K \setminus M_{K,\infty},\ \|s\|_v \leqslant 1\}$$

forms an Euclidean lattice inside $\bigoplus_{\sigma \in M_{K,\infty}} E \otimes_K K_\sigma$. It can be shown that one has (see for example [**10**] Proposition 4.6)

$$\nu_{\min}^{\mathrm{BV}}(\overline{E}) \geqslant \nu_{\min}(\overline{\mathcal{E}}). \tag{4.2}$$

Moreover, the inequality

$$\widehat{\mu}_{\min}(\overline{\mathcal{E}}) \leqslant \widehat{\mu}_{\min}(\overline{E}) + \frac{\ln |\Delta_K|}{[K : \mathbb{Q}]} \tag{4.3}$$

holds, provided that $\overline{E}$ is pure. This can for example be deduced from [**3**, (2.1.13)]. Therefore Banaszczyk's transference theorem applied to $\overline{\mathcal{E}}$ leads to (see **[10]** Corollary 4.8) the following inequality for any pure Hermitian adelic vector bundle

$$\widehat{\mu}_{\min}(\overline{E}) - \widehat{\nu}_{\min}^{\mathrm{BV}}(\overline{E}) \leqslant \ln([K:\mathbb{Q}]\operatorname{rk}_K(E)) + \frac{\ln|\Delta_K|}{[K:\mathbb{Q}]}.$$

In the case where the Hermitian adelic vector bundle $\overline{E}$ is not pure, the inequality (4.2) remains true. However, the inequality (4.3) should be corrected by the "*default of purity*" introduced in [**15**, §2.1.1].

4.2. Arakelov geometry method. — Let $\overline{E}$ be a Hermitian adelic vector bundle on $\operatorname{Spec} K$. The Arakelov theory permits to consider the function $-\widehat{\deg}_{\overline{E}}(\cdot)$ on $(E \otimes_K K^{\mathrm{a}}) \setminus \{0\}$ as an Arakelov height function on the set of algebraic points of $\mathbb{P}(E^\vee)$. Denote by $\mathcal{O}_{E^\vee}(1)$ be the tautological line bundle on $\mathbb{P}(E^\vee)$.

For any place $v \in M_K$, let $\mathbb{P}(E^\vee)_v^{\mathrm{an}}$ be the analytic space (in the sense of Berkovich if v is non-Archimedean) associated with

$$\mathbb{P}(E^\vee) \times_{\operatorname{Spec} K} \operatorname{Spec} K_v \cong \mathbb{P}(E^\vee_{K_v}).$$

Recall that each point x of $\mathbb{P}(E^\vee)_v^{\mathrm{an}}$ corresponds to a pair $(P(x), |\cdot|_x)$, where $P(x)$ is a scheme point of $\mathbb{P}(E^\vee_{K_v})$, and $|\cdot|_x$ is an absolute value on the residue field of $P(x)$ which extends the absolute value $|\cdot|_v$ on K_v. We denote by $\widehat{\kappa}(x)$ the completion of the residue field of $P(x)$ with respect to the absolute value $|\cdot|_x$. By abuse of notation, the continuous extension of $|\cdot|_x$ on $\widehat{\kappa}(x)$ is still denoted by $|\cdot|_x$. By the functorial interpretation of the scheme $\mathbb{P}(E^\vee_{K_v})$, the scheme point $P(x)$ corresponds to a surjective $\widehat{\kappa}(x)$-linear map

$$E^\vee_{K_v} \otimes_{K_v} \widehat{\kappa}(x) \longrightarrow P(x)^*(\mathcal{O}_{E^\vee}(1)_{K_v}) \otimes_{K_v} \widehat{\kappa}(x).$$

The extension of the dual norm $\|\cdot\|_{v,*}$ induces by quotient a norm on the one-dimensional vector space

$$\mathcal{O}_{E^\vee}(1)(x) := P(x)^*(\mathcal{O}_{E^\vee}(1)_{K_v}) \otimes_{K_v} \widehat{\kappa}(x),$$

denoted by $|\cdot|_{\mathrm{FS}_v}(x)$ and called the *Fubini-Study norm* on $\mathcal{O}_{E^\vee}(1)(x)$ associated with $\|\cdot\|_v$. The Fubini-Study norms $\big(|\cdot|_{\mathrm{FS}_v}(x)\big)_{x\in\mathbb{P}(E^\vee)_v^{\mathrm{an}}}$ form a continuous metric on $\mathcal{O}_{E^\vee}(1)_{K_v}$, called the *Fubini-Study metric* associated with $\|\cdot\|_v$. The line bundle $\mathcal{O}_{E^\vee}(1)$ equipped with the family of Fubini-Study metrics indexed by $v \in M_K$ forms an *adelic line bundle* on $\mathbb{P}(E^\vee)$ in the sense of Zhang [**26**].

Let X be an integral projective scheme over $\operatorname{Spec} K$ and $\overline{L} = (L, (\varphi_v)_{v\in M_K})$ be an adelic line bundle on X. Recall that L is an invertible $\mathcal{O}_X$-module, and each φ_v is a continuous metric on L_v, the pull-back of L on $X_v = X \times_{\operatorname{Spec} K} \operatorname{Spec} K_v$ by the projection morphism. For any closed point y of X, the metric structure of $\overline{L}$ induces a structure of adelic line bundle on $y^*(L)$ (viewed as a one-dimensional vector space over the residue field of y). The *Arakelov height* of y with respect to $\overline{L}$ is defined as the Arakelov degree of $\overline{y^*(L)}$ and is denoted by $h_{\overline{L}}(y)$. Moreover, the linear series $H^0(X,L)$ is naturally equipped with supremum norms $\|\cdot\|_{\varphi_v}$, where

$$s \in H^0(X,L)\otimes_K K_v = H^0(X_v,L_v), \quad \|s\|_{\varphi_v} := \sup_{x\in X_v^{\mathrm{an}}} |s|_{\varphi_v}(x).$$

The datum

$$\overline{H^0(X,L)} = (H^0(X,L),(\|\cdot\|_{\varphi_v})_{v\in M_K})$$

forms actually an adelic vector bundle on $\operatorname{Spec} K$.

The following conjecture relates the essential minimum of the height function to an invariant of the metrised graded linear series of $\overline{L}$.

Conjecture 4.1. — *Let X be an integral projective scheme over* $\operatorname{Spec} K$ *and $\overline{L}=(L,(\varphi_v)_{v\in M_K})$ be an adelic line bundle on X. Assume that L is big and the metrics φ_v are semi-positive, then the following equality holds:*

$$\sup_{Y\subsetneq X}\ \inf_{y\in(X\setminus Y)(K^{\mathrm{a}})} h_{\overline{L}}(y) = \lim_{N\to+\infty}\frac{\widehat{\mu}_{\max}(H^0(X,L^{\otimes N}),(\|\cdot\|_{\varphi_v^{\otimes N}})_{v\in M_K})}{N}, \tag{4.4}$$

where Y runs over the set of all strict closed subscheme of X, and for any non-zero adelic vector bundle $\overline{E}$ on $\operatorname{Spec} K$, *$\widehat{\mu}_{\max}(\overline{E})$ denotes the supremum of slopes of non-zero adelic vector subbundles of $\overline{E}$.*

Remark 4.2. — The left-hand side of the formula (4.4) is a classic invariant of height function, called the *essential minimum* of $\overline{L}$, which is denoted by $\operatorname{ess.min.}(\overline{L})$. The limit on the right-hand side of the formula (4.4) is called the *asymptotic maximal slope* of $\overline{L}$ (see [**7**, §4.2] for the proof of its existence). It can be shown that the essential minimum is always bounded from below by the asymptotic maximal slope. The equality between these invariants is closely related to the equidistribution problem of algebraic points of small height in an arithmetic projective variety. We refer the readers to [**8**, §5.2] for more details.

Let y be a closed point of $\mathbb{P}(E^\vee)$, which corresponds to a one-dimensional quotient space of $E^\vee\otimes_K K^{\mathrm{a}}$, or, by duality, to a one-dimensional vector subspace of $E\otimes_K K^{\mathrm{a}}$. Note that the *height* of y with respect to the adelic line bundle $\overline{\mathcal{O}_{E^\vee}(1)}$ is equal to $-\widehat{\deg}(s_y)$, where s_y is an arbitrary non-zero element of $E\otimes_K K^{\mathrm{a}}$ which spans the one-dimensional vector subspace corresponding to y. Therefore $-\nu_{\min}^{\mathrm{abs}}(\overline{E})$ identifies with the essential infimum of heights of closed points of $\mathbb{P}(E^\vee)$ avoiding linear closed subschemes of codimension 1 spanned by non-zero vectors of $E\otimes_K K^{\mathrm{a}}$ of height $\geqslant -\nu_{n-1}^{\mathrm{abs}}(\overline{E})$, where n is the dimension of E over K. Therefore, one has

$$\operatorname{ess.min.}(\overline{\mathcal{O}_{E^\vee}(1)}) \geqslant -\nu_{\min}^{\mathrm{abs}}(\overline{E}). \tag{4.5}$$

In [**3**], the supremum norm of the Fubini-Study metric has been compared with the symmetric product norm. For any $v\in M_K$ and any $N\in\mathbb{N}$, we denote by $\|\cdot\|_{\mathrm{FS}_v^{\otimes N}}$ the supremum norm on

$$H^0(\mathbb{P}(E^\vee), \mathcal{O}_{E^\vee}(N)) \otimes_K K_v \cong \mathrm{Sym}^N(E_{K_v}^\vee)$$

associated with the N-th tensor power of the Fubini-Study metric, and by $\|\cdot\|_{\mathrm{Sym}_v^N}$ the N-th symmetric power of the norm $\|\cdot\|_{v,*}$. By **[3]** Lemma 4.3.6 and Corollary 1.4.3, the following inequality holds for any Archimedean places σ:

$$\|\cdot\|_{\mathrm{Sym}_\sigma^N} \leqslant \binom{N+n-1}{n-1}^{\frac{1}{2}} \|\cdot\|_{\mathrm{FS}_\sigma^{\otimes N}},$$

which leads to

$$\widehat{\mu}_{\max}(\mathrm{Sym}^N(E^\vee), (\|\cdot\|_{\mathrm{FS}_v^{\otimes N}})_{v\in M_K}) \leqslant \widehat{\mu}_{\max}(\mathrm{Sym}^N(E^\vee), (\|\cdot\|_{\mathrm{Sym}_v^N})_{v\in M_K}) + \frac{1}{2}\ln\binom{N+n-1}{n-1}.$$

Moreover, by [**17**, Theorem 6.1] (see also §8.2 *loc. cit.*), one has

$$\widehat{\mu}_{\max}(\mathrm{Sym}^N(E^\vee), (\|\cdot\|_{\mathrm{Sym}_v^N})_{v\in M_K}) \leqslant N\Big(\widehat{\mu}_{\max}(\overline{E}^\vee) + \sum_{i=1}^{n-1}\frac{1}{i}\Big) + o(N).$$

Therefore, we obtain

$$\lim_{N\to+\infty} \frac{\widehat{\mu}_{\max}(\mathrm{Sym}^N(E^\vee), (\|\cdot\|_{\mathrm{FS}_v^{\otimes N}})_{v\in M_K})}{N} \leqslant -\widehat{\mu}_{\min}(\overline{E}) + \sum_{i=1}^{n-1}\frac{1}{i}. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain the following result.

Theorem 4.3. — *Assume that Conjecture 4.1 is true. For any Hermitian adelic vector bundle $\overline{E}$ over* $\mathrm{Spec}\,K$*, the following inequality holds*

$$\widehat{\mu}_{\min}(\overline{E}) \leqslant \widehat{\nu}_{\min}^{\mathrm{abs}}(\overline{E}) + \sum_{i=1}^{n-1}\frac{1}{i}, \tag{4.7}$$

where $n = \mathrm{rk}_{\mathbb{Z}}(E)$.

Remark 4.4. — It is plausible that the method of [**11**, §4] leads to a finer inequality (4.6) in replacing $\sum_{i=1}^{n-1}\frac{1}{i}$ by $\frac{1}{2}\ln(n)$.

4.3. Application to the transference problem. — We have observed that a transference inequality for the sum of the last minimum of a Hermitian adelic vector bundle and the first minimum of its dual leads to a comparison result between successive minima and successive slope. In this subsection, we explain how to deduce a general transference inequality from the comparison between minima and slopes.

Proposition 4.5. — *Let n be an integer such that $n \geqslant 1$. Let $\natural \in \{\mathrm{BV}, \mathrm{RT}, \mathrm{abs}\}$. For any Hermitian adelic vector bundle $\overline{E}$ of rank n on* $\operatorname{Spec} K$*, one has*

$$\widehat{\nu}_i^{\natural}(\overline{E}) + \widehat{\nu}_{n+1-i}^{\natural}(\overline{E}^\vee) \geqslant -2\delta^{\natural}(n). \tag{4.8}$$

Proof. — By definition one has

$$\widehat{\nu}_i^{\natural}(\overline{E}) \geqslant \widehat{\mu}_i(\overline{E}) - \delta^{\natural}(n) \quad \text{and} \quad \widehat{\nu}_{n+1-i}^{\natural}(\overline{E}) \geqslant \widehat{\mu}_{n+1-i}(\overline{E}) - \delta^{\natural}(n).$$

Taking the sum of the two inequality we obtain

$$\widehat{\nu}_i^{\natural}(\overline{E}) + \widehat{\nu}_{n+1-i}^{\natural}(\overline{E}^\vee) \geqslant \widehat{\mu}_i(\overline{E}) + \widehat{\mu}_{n+1-i}(\overline{E}^\vee) - 2\delta^{\natural}(n) = -2\delta^{\natural}(n).$$

□

The interest of the above proposition is to deduce, from an upper bound of $\delta^{\natural}(n)$, a lower bound for $\widehat{\nu}_i^{\natural}(\overline{E}) + \widehat{\nu}_{n+1-i}^{\natural}(\overline{E}^\vee)$. In particular, we propose the following conjecture.

Conjecture 4.6. — *Let n be an integer such that $n \geqslant 1$ and $\overline{E}$ be a Hermitian adelic vector bundle on* $\operatorname{Spec} K$. *For any $i \in \{1, \ldots, n\}$ one has*

$$\widehat{\nu}_i^{\mathrm{abs}}(\overline{E}) + \widehat{\nu}_{n+1-i}^{\mathrm{abs}}(\overline{E}^\vee) \geqslant -\ln(n).$$

This conjecture is an analogue over $\mathbb{Q}^{\mathrm{a}}$ of Banaszczyk's transference theorem. It can for example be deduced from Conjecture 4.1 and the following conjectural inequality

$$\limsup_{N \to +\infty} \frac{\widehat{\mu}_{\max}(\mathrm{Sym}^N(\overline{E}^\vee))}{N} \leqslant \widehat{\mu}_{\max}(\overline{E}) + \frac{1}{2}\ln(n).$$

Such observation opens a new way to study the transference problem by Arakelov geometry instead of classic geometry of numbers.

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