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ON ISOPERIMETRIC INEQUALITY IN ARAKELOV GEOMETRY

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ABSTRACT. — We establish an isoperimetric inequality in an integral form and deduce a relative version of Brunn-Minkowski inequality in the Arakelov geometry setting.

RÉSUMÉ (*Sur l'inégalité isopérimétrique en géométrie d'Arakelov*). — On établit une inégalité isopérimétrique sous une forme d'intégration dans le cadre de géométrie d'Arakelov et en déduit une version relative de l'inégalité de Brunn-Minkowski dans le même cadre.

1. Introduction

The isoperimetric inequality in Euclidean geometry asserts that, for any convex body Δ in $\mathbb{R}^d$, one has

$$\text{vol}(\partial\Delta)^d \geq d^d \,\text{vol}(B)\,\text{vol}(\Delta)^{d-1}, \tag{1}$$

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where B denotes the closed unit ball in $\mathbb{R}^d$. From the point of view of convex geometry, the isoperimetric inequality can be deduced from the Brunn-Minkowski inequality: for two Borel subsets A_1 and A_2 in $\mathbb{R}^d$, one has

$$\operatorname{vol}(A_0+A_1)^{1/d} \geq \operatorname{vol}(A_0)^{1/d} + \operatorname{vol}(A_1)^{1/d}, \tag{2}$$

where

$$A_0 + A_1 := \{x+y | x \in A_0, y \in A_1\}$$

is the Minkowski sum of A_0 and A_1. The proof consists of taking $A_0 = \Delta$ and $A_1 = \varepsilon B$ in (2) with $\varepsilon > 0$ and letting ε tend to 0. We refer readers to [35] for a presentation on the history of the isoperimetric inequality and to page 1190 of *loc. cit.* for more details on how to deduce (1) from (2). The same method actually leads to a lower bound for the mixed volume of convex bodies:

$$\operatorname{vol}_{d-1,1}(\Delta_0, \Delta_1)^d \geq \operatorname{vol}(\Delta_0)^{d-1} \cdot \operatorname{vol}(\Delta_1), \tag{3}$$

where Δ_0 and Δ_1 are two convex bodies in $\mathbb{R}^d$ and $\operatorname{vol}_{d-1,1}(\Delta_0, \Delta_1)$ is the mixed volume of index $(d-1,1)$, which is equal to

$$\lim_{\varepsilon \to 0+} \frac{\operatorname{vol}(\Delta_0 + \varepsilon \Delta_1) - \operatorname{vol}(\Delta_0)}{\varepsilon d}.$$

We refer readers to the work of Minkowski [31] for the notion of mixed volumes in convex geometry. See [8, § 7.29] for more details.

Note that (3) is one of the inequalities of Alexandrov-Fenchel type for mixed volumes, which is actually equivalent to Brunn-Minkowski inequality (see for example [37, § 7.2] for a proof). Note that the above inequalities in convex geometry are similar to some inequalities of intersection numbers in algebraic geometry. By using toric varieties, Teissier [38] and Khovanskii [13, § 4.27] have given proofs of the Alexandrov-Fenchel inequality by using the Hodge index theorem.

In the arithmetic geometry setting, Bertrand [6, § 1.2] has established a lower bound for the height function on an arithmetic variety, and interpreted it as an arithmetic analogue of the isoperimetric inequality. In [15], the author has proposed the notion of positive intersection product in Arakelov geometry and proved an analogue of the isoperimetric inequality in the form of (3), by using the arithmetic Brunn-Minkowski inequality established by Yuan [40].

The purpose of this article is to propose a relative version of the arithmetic isoperimetric inequality and Brunn-Minkowski inequality as follows by taking into account the relative structure of arithmetic varieties with respect to an arithmetic curve. We refer to Theorems 3.5, 3.3, 4.1 and 4.5 for the proof and for various refined forms of the statement.

THEOREM 1.1. — *Let K be a number field and X be a geometrically integral projective scheme of dimension $d \geq 1$ over* $\operatorname{Spec} K$. *Let $\overline{D}_0$ and $\overline{D}_1$ be nef* *adelic $\mathbb{R}$-Cartier divisors on X such that D_0 and D_1 are big. Then one has*

$$(4)\quad (d+1)\widehat{\deg}(\overline{D}_0^d\cdot\overline{D}_1)\geq d\left(\frac{\deg(D_1^d)}{\deg(D_0^d)}\right)^{1/d}\widehat{\deg}(\overline{D}_0^{d+1})+\frac{\deg(D_0^d)}{\deg(D_1^d)}\widehat{\deg}(\overline{D}_1^{d+1}).$$

If $(\overline{D}_i)_{i=1}^n$ is a family of nef adelic $\mathbb{R}$-Cartier divisors such that $D_1,\ldots,D_n$ are big, then one has

$$(5)\qquad \frac{\widehat{\deg}((\overline{D}_1+\cdots+\overline{D}_n)^{d+1})}{\deg((D_1+\cdots+D_n)^d)}\geq\varphi(D_1,\ldots,D_n)^{-1}\sum_{i=1}^n\frac{\widehat{\deg}(\overline{D}_i^{d+1})}{\deg(D_i^d)},$$

where

$$(6)\qquad \varphi(D_1,\ldots,D_n):=d+1-d\frac{\deg(D_1^d)^{1/d}+\cdots+\deg(D_n^d)^{1/d}}{\deg((D_1+\cdots+D_n)^d)^{1/d}}.$$

Compared to the direct arithmetic analogue of the Brunn-Minkowski inequality (see [40, Theorem B]), the inequality (5) distinguishes the contribution of the geometric structure of the $\mathbb{R}$-Cartier divisors $D_1,\ldots,D_n$.

In the particular case where $d=2$ (that is, where X is an arithmetic surface), the inequality (4) becomes a form of the arithmetic Hodge index inequality

$$2\widehat{\deg}(\overline{D}_0\cdot\overline{D}_1)\geq\frac{\deg(D_1)}{\deg(D_0)}\widehat{\deg}(\overline{D}_0^2)+\frac{\deg(D_0)}{\deg(D_1)}\widehat{\deg}(\overline{D}_1^2),$$

established in [17, Theorem 4.12], which is equivalent to the arithmetic Hodge index theorem of Faltings [19] and Hriljac [24], since the above inequality is equivalent to

$$\left(\frac{\overline{D}_0}{\deg(D_0)}-\frac{\overline{D}_1}{\deg(D_1)}\right)^2\leq 0.$$

We refer readers to [17, Corollary 4.14 and Remark 4.15] for a comparison of different forms of the statement. Similarly to [17], we also use the interpretation of the arithmetic self-intersection number of a nef and big adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ as the integral of a concave function on the Okounkov body $\Delta(D)$ of the $\mathbb{R}$-Cartier divisor D, which is a convex body in $\mathbb{R}^d$. However the proof of Theorem 1.1 follows a strategy which is different from the way indicated in [17]. In fact, in [17] the author has introduced for any couple (Δ_1,Δ_2) of convex bodies in $\mathbb{R}^d$, a number $\rho(\Delta_1,\Delta_2)$ (called the correlation index of Δ_1 and Δ_2) which measures the degree of uniformity in the Minkowski sum $\Delta_1+\Delta_2$ of the sum of two uniform random variables[1] valued in Δ_1 and Δ_2, respectively.

1. For any convex body $\Delta\subset\mathbb{R}^d$, a Borel probability measure on $\mathbb{R}^d$ is called the *uniform distribution* on Δ if it is absolutely continuous with respect to the Lebesgue measure, and the corresponding Radon-Nikodym density is $1/\operatorname{vol}(\Delta)$, where $\operatorname{vol}(\Delta)$ is the Lebesgue measure of Δ; a random variable valued in $\mathbb{R}^d$ is said to be *uniformly distributed* in Δ if it follows this measure as its probability law.

The inequality

$$\frac{\widehat{\deg}((\overline{D}_1+\overline{D}_2)^{d+1})}{\mathrm{vol}(\Delta(D_1)+\Delta(D_2))} \geq \rho(\Delta(D_1),\Delta(D_2))^{-1}\left(\frac{\widehat{\deg}(\overline{D}_1^{d+1})}{\mathrm{vol}(\Delta(D_1))}+\frac{\widehat{\deg}(\overline{D}_2^{d+1})}{\mathrm{vol}(\Delta(D_2))}\right)$$

has been established for any couple $(\overline{D}_1,\overline{D}_2)$ of nef and big adelic $\mathbb{R}$-Cartier divisors on X, and it has been suggested that the estimation of the correlation index $\rho(\Delta(D_1),\Delta(D_2))$ should lead to more concrete inequalities which are similar to (5). However, the main point in this approach is to construct a suitable correlation structure between two random variables which are uniformly distributed in $\Delta(D_1)$ and $\Delta(D_2)$ such that the sum of the random variables is as uniform as possible in the Minkowski sum $\Delta(D_1)+\Delta(D_2)$. For example, we can deduce from a work of Bobkov and Madiman [7] the following uniform upper bound (where we choose independent random variables) (see [17, Proposition 2.9])

$$\rho(\Delta(D_1),\Delta(D_2)) \leq \binom{2d}{d}.$$

This upper bound is larger than $\varphi(D_1,D_2)$, the latter being clearly bounded from above by $d+1$.

The strategy of this article is inspired by the works of Knothe [28] and Brenier [11, 12] on measure preserving diffeomorphisms between two convex bodies (see also the works of Gromov [22], Alesker, Dar and Milman [2] for more developments of this method and for applications in Alexandrov-Fenchel type inequalities in the convex geometry setting, and the memoir of Barthe [3] for diverse applications of this method in functional inequalities). Given a couple (Δ_0,Δ_1) of convex bodies in $\mathbb{R}^d$, one can construct a C^1 diffeomorphism $f:\Delta_0\to\Delta_1$ which transports the uniform probability measure of Δ_0 to that of Δ_1; that is, the determinant of the Jacobian J_f is constant on the interior of Δ_0. This diffeomorphism is not unique: in the construction of Knothe, the Jacobian J_f is upper triangular, while in the construction of Brenier, J_f is symmetric and positive definite.

If Z_0 is a random variable which is uniformly distributed in Δ_0, then $Z_1 := f(Z_0)$ is uniformly distributed in Δ_1. One may expect that the random variable Z_0+Z_1 follows a probability law which is close to the uniform probability measure on $\Delta_0+\Delta_1$. In fact, the random variable Z_0+Z_1 can also be expressed as $Z_0+f(Z_0)$. Its probability law identifies with the direct image of the uniform probability measure on Δ_0 by the map $\mathrm{Id}+f$, admitting $\mathrm{Id}+J_f$ as its Jacobian, the determinant of which can be estimated in terms of the determinant of J_f. In the case where $\mathrm{Id}+f$ is injective (for example the Knothe map), this lower bound leads to the following upper bound for the correlation index

$$\rho(\Delta_0,\Delta_1) \leq \frac{\mathrm{vol}(\Delta_0+\Delta_1)}{(\mathrm{vol}(\Delta_0)^{1/d}+\mathrm{vol}(\Delta_1)^{1/d})^d}. \tag{7}$$

By this method we obtain a weaker version of inequality (5) in the case where $n=2$ by replacing $\varphi(D_1,D_2)$ with

$$\frac{\mathrm{vol}(D_1+D_2)}{(\mathrm{vol}(D_1)^{1/d}+\mathrm{vol}(D_2)^{1/d})^d}.$$

This function is, in general, not bounded when D_1 and D_2 vary.

The main idea of the article is to use an infinitesimal variant of the above argument. Instead of considering the map $\mathrm{Id}+f:\Delta_0\rightarrow\Delta_0+\Delta_1$, we consider $\mathrm{Id}+\varepsilon f:\Delta_0\rightarrow\Delta_0+\varepsilon\Delta_1$ for $\varepsilon>0$ sufficiently small, and use it to establish an isoperimetric inequality in an integral form as follows (see Theorem 3.1 *infra*).

Theorem 1.2. — *Let G_0 and G_1 be two Borel functions on Δ_0 and Δ_1, respectively. We assume that they are integrable with respect to the Lebesgue measure. Suppose given, for any $\varepsilon\in[0,1]$, an almost everywhere non-negative Borel function H_ε on $\Delta_0+\varepsilon\Delta_1$ such that*

$$\forall\,(x,y)\in\Delta_0\times\Delta_1,\quad H_\varepsilon(x+\varepsilon y)\geq G_0(x)+\varepsilon G_1(y).$$

Then the following inequality holds

$$\begin{aligned}&\liminf_{\varepsilon\rightarrow 0+}\frac{\int_{\Delta_0+\varepsilon\Delta_1}H_\varepsilon(z)\mathrm{d}z-\int_{\Delta_0}G_0(x)\mathrm{d}x}{\varepsilon}\\&\geq d\left(\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)}\right)^{1/d}\int_{\Delta_0}G_0(x)\mathrm{d}x+\frac{\mathrm{vol}(\Delta_0)}{\mathrm{vol}(\Delta_1)}\int_{\Delta_1}G_1(y)\mathrm{d}y.\end{aligned}$$

By this method we obtain a relative form of the arithmetic isoperimetric inequality as in (4) and then deduce the arithmetic relative Brunn-Minkowski inequality (5) following the classic procedure of deducing the Brunn-Minkowski inequality from the isoperimetric inequality. Note that this does not signify that we improve inequality (7) by replacing the right-hand side of the inequality with

$$d+1-d\frac{\mathrm{vol}(\Delta_0)^{d/1}+\mathrm{vol}(\Delta_1)^{1/d}}{\mathrm{vol}(\Delta_0+\Delta_1)^{1/d}}.$$

For example, it remains an open question to determine whether the correlation index $\rho(\Delta_0,\Delta_1)$ is always bounded from above by $d+1$.

Finally, I would like to cite several refinements of the Brunn-Minkowski inequality in convex geometry, where the results are also expressed in a relative form similarly to (5), either with respect to an orthogonal projection on a hyperplane [23] or in terms of a comparison between the volume and the mixed volume [20] in the style of Bergstrom's inequality [4]. It is not excluded that the method presented in this article will bring new ideas to the research efforts in these directions.

The article is organized as follows. In the second section, we recall the notation and basic facts about adelic $\mathbb{R}$-Cartier divisors. In the third section, we prove a relative version of isoperimetric inequality in convex geometry and deduce the arithmetic isoperimetric inequality (4). In the fourth and last section, we prove the relative arithmetic Brunn-Minkowski inequality (5).

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2. Reminder on adelic divisors

Throughout this article, K denotes a field. Let X be an integral projective scheme over K and d be its Krull dimension.

2.1. $\mathbb{R}$-Cartier divisors. — In this subsection, we recall some notions and facts about $\mathbb{R}$-Cartier divisors on a projective variety.

2.1.1. Denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X and by $\mathrm{Div}^+(X)$ the sub-semigroup of $\mathrm{Div}(X)$ of effective divisors. Let $\mathrm{Div}(X)_{\mathbb{R}}$ be the real vector space $\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{R}$, the elements of which are called $\mathbb{R}$-Cartier divisors. An $\mathbb{R}$-Cartier divisor D is said to be *effective* if it belongs to the positive cone generated by effective Cartier divisors on X. We use the expression $D\geq 0$ to denote the effectivity of an $\mathbb{R}$-Cartier divisor D.

2.1.2. Let D be an $\mathbb{R}$-Cartier divisor on X. We denote by $H^0(D)$ the set

$$\{f\in K(X)^{\times}\,|\,\mathrm{div}(f)+D\geq 0\}\cup\{0\},$$

where $K(X)$ is the field of all rational functions on X, and $\mathrm{div}(f)$ denotes the principal divisor associated with the rational function f. This is a K-vector subspace of finite rank of $K(X)$. We denote by $h^0(D)$ its rank over K. Recall that the *volume* of D is defined as

$$\mathrm{vol}(D):=\limsup_{n\to+\infty}\frac{h^0(nD)}{n^d/d!}.$$

If $\mathrm{vol}(D)>0$, then the $\mathbb{R}$-Cartier divisor D is said to be *big*. The big $\mathbb{R}$-Cartier divisors form an open cone in $\mathrm{Div}(X)_{\mathbb{R}}$, denoted by $\mathrm{Big}_{\mathbb{R}}(X)$.

2.1.3. A Cartier divisor D is said to be *ample* if the associated invertible sheaf $\mathcal{O}(D)$ is ample. An $\mathbb{R}$-Cartier divisor is said to be *ample* if it can be written as a linear combination of ample Cartier divisors with positive coefficients. An $\mathbb{R}$-Cartier divisor D on X is said to be *numerically effective* (*nef*) if, for any ample $\mathbb{R}$-Cartier divisor D', the sum $D+D'$ is ample. Denote by $\mathrm{Nef}_{\mathbb{R}}(X)$ the cone of nef $\mathbb{R}$-Cartier divisors on X.

2.1.4. Recall that the function of self-intersection number $D \mapsto \deg(D^d)$ is a homogeneous polynomial of degree d on the vector space $\mathrm{Div}(X)_{\mathbb{R}}$. Its polar form

$$(D_1,\ldots,D_d) \in \mathrm{Div}(X)_{\mathbb{R}}^d \longmapsto \deg(D_1\cdots D_d)$$

is the function of the intersection number. Note that the volume of a nef $\mathbb{R}$-Cartier divisor D coincides with the self-intersection number of D. In particular, the volume function is a homogeneous polynomial of degree d on the nef cone $\mathrm{Nef}_{\mathbb{R}}(X)$.

2.1.5. Let D be an $\mathbb{R}$-Cartier divisor on X. We call a *linear system* of D any K-vector subspace of $H^0(D)$. We call a *graded linear series* of D any $\mathbb{N}$-graded sub-K-algebra of $V_\bullet(D) := \bigoplus_{n\in\mathbb{N}} H^0(nD)$. If $V_\bullet = \bigoplus_{n\in\mathbb{N}} V_n$ is a graded linear series of D, its *volume* is defined as

$$\mathrm{vol}(V_\bullet) := \limsup_{n\to\infty} \frac{\dim_K(V_n)}{n^d/d!}.$$

Therefore the volume of the total graded linear series $V_\bullet(D)$ is equal to the volume of the $\mathbb{R}$-Cartier divisor D.

Following [30, Definition 2.9], we say that a graded linear series $V_\bullet$ of an $\mathbb{R}$-Cartier divisor D *contains an ample* $\mathbb{R}$-*Cartier divisor* if there exists an ample $\mathbb{R}$-Cartier divisor A such that $V_\bullet(A) \subset V_\bullet$ (see also [17, Remark 3.2] for some equivalent forms). This condition implies that the volume of $V_\bullet$ is > 0.

By the works of Lazarsfeld and Mustaţă [30] and Kaveh and Khovanskii [27, 26] (see also the work of Cutkosky [18] which allows relaxing the assumption on the existence of a regular rational point), to each graded linear series $V_\bullet$ of some $\mathbb{R}$-Cartier divisor, which contains an ample $\mathbb{R}$-Cartier divisor, we can attach a convex body $\Delta(V_\bullet)$ (called the *Newton-Okounkov body* of $V_\bullet$), such that

$$\mathrm{vol}(V_\bullet) = d!\,\mathrm{vol}(\Delta(V_\bullet)),$$

where $\mathrm{vol}(\Delta(V_\bullet))$ denotes the Lebesgue measure of the convex body $\Delta(V_\bullet)$. We refer readers to [30, Theorem 2.13] for more details.

2.1.6. Let $V_\bullet$ and $V'_\bullet$ be graded linear series of two $\mathbb{R}$-Cartier divisors D and D' respectively. Let $W_\bullet$ be a graded linear series of $D+D'$ such that

$$\forall\, n\in\mathbb{N},\quad \{fg \,|\, f\in V_n,\ g\in V'_n\} \subset W_n.$$

Assume that the graded linear series $V_\bullet$ and $V'_\bullet$ contain ample $\mathbb{R}$-Cartier divisors. Then the graded linear series $W_\bullet$ also contains an ample $\mathbb{R}$-Cartier divisor. Moreover, one has

$$\Delta(V_\bullet) + \Delta(V'_\bullet) \subset \Delta(W_\bullet).$$

Therefore the Brunn-Minkowski theorem (in the classic convex geometry setting) leads to

$$\mathrm{vol}(W_\bullet)^{1/d} \geq \mathrm{vol}(V_\bullet)^{1/d} + \mathrm{vol}(V'_\bullet)^{1/d}. \tag{8}$$

2.2. Adelic $\mathbb{R}$-Cartier divisors. — In this subsection, we recall some notions and facts about adelic $\mathbb{R}$-Cartier divisors. The references are [21, 34]. We assume that K is a number field. Let M_K be the set of all places of K. For any place $v \in M_K$, let $|\cdot|_v$ be the absolute value on K in the equivalence class v which extends either the usual absolute value on $\mathbb{Q}$ or certain p-adic absolute value (such that $|p|_v = p^{-1}$), where p is a prime number. Denote by K_v the completion of the field K with respect to the topology corresponding to the place v, on which the absolute value $|\cdot|_v$ extends in a unique way.

2.2.1. Let $\pi : X \to \operatorname{Spec} K$ be a geometrically integral K-scheme. For any $v \in M_K$, let X_v^{an} be the Berkovich analytic space associated with the K_v-scheme $X_v := X \otimes_K K_v$. We denote by $j_v : X_v^{\mathrm{an}} \to X_v$ the map which sends any element $x \in X_v^{\mathrm{an}}$ to its underlying point in X_v. The most coarse topology on X_v^{an} which makes the map j_v continuous is called the *Zariski topology* on X_v^{an}.

Berkovich [5] defines another topology on X_v^{an} which is finer than the Zariski topology. If U is a Zariski open subset of X_v and f is a regular function on U, then for each point $x \in U^{\mathrm{an}} := j_v^{-1}(U)$, the regular function f defines by reduction an element $f(x)$ in the residue field of $j_v(x)$. Note that, by the construction of the Berkovich analytic space X_v^{an}, this residue field is equipped with an absolute value (depending on x) which extends $|\cdot|_v$. We denote by $|f|_v(x)$ the absolute value of $f(x)$. Thus we obtain a real-valued function $|f|_v$ on $j_v^{-1}(U)$. The *Berkovich topology* is then defined as the most coarse topology on X_v^{an} which makes continuous the map j_v and all functions of the form $|f|_v$, where f is a regular function on some Zariski open subset of X_v. The set X_v^{an} equipped with the Berkovich topology is a compact Hausdoff space (see [5, Theorem 3.4.8]).

2.2.2. Let v be a place of K. We denote by $\mathcal{C}^0_{X_v^{\mathrm{an}}}$ the sheaf of continuous real-valued functions on the topological space X_v^{an} (with the Berkovich topology). For any Berkovich open subset V of X_v^{an}, denote by $C^0(V)$ the set of all sections of $\mathcal{C}^0_{X_v^{\mathrm{an}}}$ over V. It is a vector space over $\mathbb{R}$. Let $\widehat{C}^0(X_v^{\mathrm{an}})$ be the colimit of the vector spaces $C^0(U^{\mathrm{an}})$, where U runs over the (filtered) ordered set of all non-empty Zariski open subsets of X_v. Note that any non-empty Zariski open subset of X_v^{an} is dense in X_v^{an} for the Berkovich topology (see [5, Proposition 3.4.5]). Therefore, for any non-empty Zariski open subset U of X_v, the natural map $C^0(U^{\mathrm{an}}) \to \widehat{C}^0(X_v^{\mathrm{an}})$ is injective. If an element in $\widehat{C}^0(X_v^{\mathrm{an}})$ belongs to the image of this map, we say that it *extends* to a continuous function on U^{an}.

If f is a rational function on X_v^{an}, then it identifies with a regular function on some non-empty Zariski open subset U of X_v. Therefore the function $|f|_v$ determines an element in $\widehat{C}^0(X_v^{\mathrm{an}})$. If f is non-zero, by shrinking the Zariski open set U, we may assume that $f(x)\neq 0$ for any $x\in U$. Therefore the continuous function $\log|f|_v$ on U^{an} also determines an element $\widehat{C}^0(X_v^{\mathrm{an}})$, which we still denote by $\log|f|_v$ by abuse of notation. Thus we obtain a group homomorphism from $K(X_v)^\times$ (where $K(X_v)$ denotes the field of all rational functions on X_v) to $\widehat{C}^0(X_v)$, which induces an $\mathbb{R}$-linear homomorphism from $K(X_v)^\times_{\mathbb{R}}:=K(X_v)^\times\otimes_{\mathbb{Z}}\mathbb{R}$ to $\widehat{C}^0(X_v^{\mathrm{an}})$.

2.2.3. Let D be an $\mathbb{R}$-Cartier divisor on X. For any $v\in M_K$, it induces by extension of scalars an $\mathbb{R}$-Cartier divisor D_v on X_v. We say that an element $f\in K(X_v)^\times_{\mathbb{R}}$ defines D_v locally on a Zariski open subset U of X_v if one can write D_v as $\lambda_1D_1+\cdots+\lambda_nD_n$ and f as $f=f_1^{\lambda_1}\cdots f_n^{\lambda_n}$, where $D_1,\ldots,D_n$ are Cartier divisors on X_v and $f_1,\ldots,f_n$ are elements of $K(X_v)^\times$ and $\lambda_1,\ldots,\lambda_n$ are real numbers, such that f_i defines D_i on U for each i. We call a *v-Green function* of D any element $g_v\in\widehat{C}^0(X_v^{\mathrm{an}})$ such that, for any element $f\in K(X_v)^\times_{\mathbb{R}}$ which defines D_v locally on a Zariski open subset U, the element $g_v+\log|f|_v$ extends to a continuous function on U^{an}. Note that for each element $s\in H^0(D)$, the element $|s|_v\mathrm{e}^{-g_v}\in\widehat{C}^0(X_v)$ extends to a continuous function on X_v^{an} (see [34, Proposition 2.1.3], see also [17, Remark 4.2]). Note that our choice of normalization for the Green function is different from that in [34]. Moreover, the map

$$s\longmapsto \|s\|_{g_v}:=\sup_{x\in X_v^{\mathrm{an}}}|s|_v(x)\mathrm{e}^{-g_v(x)}$$

is a norm on $H^0(D)$, which extends by continuity to a norm on $H^0(D)\otimes_K K_v$.

2.2.4. In the case where v is a non-Archimedean place of K, a typical example of v-Green function is that arising from an integral model. Let D be an $\mathbb{R}$-Cartier divisor on X. An *integral model* of (X,D) consists of a projective and flat $\mathcal{O}_K$-scheme $\mathscr{X}$ such that $\mathscr{X}_K=X$, and an $\mathbb{R}$-Cartier divisor $\mathscr{D}$ on $\mathscr{X}$ such that $\mathscr{D}|_X=D$, where $\mathcal{O}_K$ denotes the ring of algebraic integers in K.

Let x be a point in X_v^{an} and $\kappa(x)$ be the residue field of $j_v(x)$. Then $\kappa(x)$ is naturally equipped with an absolute value which extends the absolute value $|\cdot|_v$ on K_v. Let $\kappa(x)^\circ$ be the valuation ring of $\kappa(x)$. Then the valuative criterion of properness leads to a unique morphism $\mathscr{P}_x:\operatorname{Spec}\kappa(x)^\circ\to\mathscr{X}$ which extends the K-morphism $\operatorname{Spec}\kappa(x)\to X$ determined by the point x. In the case where $j_v(x)$ is outside of $\mathrm{Supp}(D_v)$, the pull-back of $\mathscr{D}$ by the morphism $\mathscr{P}_x:\operatorname{Spec}\kappa(x)^\circ\to\mathscr{X}$ is well defined, and is proportional to the divisor on $\operatorname{Spec}\kappa(x)^\circ$ corresponding to the maximal ideal of $\kappa(x)^\circ$. In other words, there exists a unique real number, which we denote by $g_{(\mathscr{X},\mathscr{D}),v}(x)$, such that

$$\mathscr{P}_x^*(\mathscr{D})=g_{(\mathscr{X},\mathscr{D}),v}(x)[\kappa(x)^{\circ\circ}],$$

where $[\kappa(x)^{\circ\circ}]$ is the Cartier divisor of $\operatorname{Spec}\kappa(x)^\circ$ defined by the maximal ideal $\kappa(x)^{\circ\circ}$ of $\kappa(x)^\circ$. Note that the element in $\widehat{C}^0(X_v^{\mathrm{an}})$ determined by the map $g_{(\mathscr{X},\mathscr{D}),v}$ is a v-Green function of D (see [34, Proposition 2.1.4]), called the v-Green function associated with the integral model $(\mathscr{X},\mathscr{D})$.

2.2.5. By *adelic* $\mathbb{R}$-*Cartier divisor* on X, we refer to any data $\overline{D}$ of the form $(D,(g_v)_{v\in M_K})$, where D is an $\mathbb{R}$-Cartier divisor on X and each g_v is a v-Green function of D. We also require that there exists an integral model $(\mathscr{X},\mathscr{D})$ of (X,D) such that $g_v=g_{(\mathscr{X},\mathscr{D}),v}$ for all but a finite number of non-Archimedean places $v\in M_K$. If D is effective and if each v-Green function g_v is non-negative (in the sense that e^{-g_v} extends to a continuous function on X_v^{an} which is bounded from above by 1), we say that the adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ is *effective*, denoted by $\overline{D}\geq 0$.

If f is an element in $K(X)_{\mathbb{R}}^{\times}$, we denote by $\operatorname{div}(f)$ the $\mathbb{R}$-Cartier divisor associated with f. For each place $v\in M_K$, let f_v be the element of $K(X_v)_{\mathbb{R}}^{\times}$ determined by f. Then the couple $(\operatorname{div}(f),(-\log|f_v|_v)_{v\in M_K})$ defines an adelic $\mathbb{R}$-Cartier divisor on X, denoted by $\widehat{\operatorname{div}}(f)$.

Note that adelic $\mathbb{R}$-Cartier divisors on $\operatorname{Spec}K$ are of the form

$$\zeta=(\mathbf{0},(\zeta_v)_{v\in M_K}),$$

where $\mathbf{0}$ denotes the zero $\mathbb{R}$-Cartier divisor on $\operatorname{Spec}K$, ζ_v are real numbers such that $\zeta_v=0$ for all but a finite number of $v\in M_K$. We denote by $\pi^*(\zeta)$ the adelic $\mathbb{R}$-Cartier divisor on X consisting of the zero $\mathbb{R}$-Cartier divisor and constant functions of value ζ_v, $v\in M_K$.

If $\overline{D}_1=(D_1,(g_{1,v})_{v\in M_K})$ and $\overline{D}_2=(D_2,(g_{2,v})_{v\in M_K})$ are two adelic $\mathbb{R}$-Cartier divisors, λ and μ are two real numbers, then

$$\lambda\overline{D}_1+\mu\overline{D}_2:=(\lambda D_1+\mu D_2,(\lambda g_{1,v}+\mu g_{2,v})_{v\in M_K})$$

is an adelic $\mathbb{R}$-Cartier divisor. Therefore the set $\widehat{\operatorname{Div}}_{\mathbb{R}}(X)$ of all adelic $\mathbb{R}$-Cartier divisors forms a vector space over $\mathbb{R}$.

If $\overline{D}$ is an adelic $\mathbb{R}$-Cartier divisor on X, the set

$$\widehat{H}^0(\overline{D}):=\{s\in H^0(D):\forall\,v\in M_K,\|s\|_{g_v}\leq 1\}$$

is finite (see Section 2.2.3 for the definition of $\|\cdot\|_{g_v}$). The *arithmetic volume* of $\overline{D}$ is defined as (see [33] and [34, § 4.3])

$$\widehat{\operatorname{vol}}(\overline{D}):=\limsup_{n\to+\infty}\frac{\log\#\widehat{H}^0(n\overline{D})}{n^{d+1}/(d+1)!}. \tag{9}$$

The adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ is said to be *big* if $\widehat{\operatorname{vol}}(\overline{D})>0$. We denote by $\widehat{\operatorname{Big}}_{\mathbb{R}}(X)$ the cone of all big adelic $\mathbb{R}$-Cartier divisors. It is an open cone in $\widehat{\operatorname{Div}}_{\mathbb{R}}(X)$ in the sense that, if $\overline{D}$ is a big adelic $\mathbb{R}$-Cartier divisor and $\overline{D}'$ is an adelic $\mathbb{R}$-Cartier divisor, then there exists $\varepsilon>0$ such that $\overline{D}+t\overline{D}'$ is big for any t such that $|t|<\varepsilon$.

2.2.6. Recall that an *adelic vector bundle* on $\operatorname{Spec} K$ is defined as any data of the form $\overline{E} = (E, (\|\cdot\|_v)_{v\in M_K})$, where E is a vector space of finite rank over K, and for any $v \in M_K$, $\|\cdot\|_v$ is a norm on $E\otimes_K K_v$, which is ultrametric if v is non-Archimedean. We also require that, for all but a finite number of places $v \in M_K$, the norm $\|\cdot\|_v$ arises from a common integral model of E, or equivalently, there exists a basis $(e_i)_{i=1}^r$ of E over K such that, for all but a finite number of $v \in M_K$, one has

$$\forall\,(\lambda_1,\ldots,\lambda_r)\in K_v^r,\quad \|\lambda_1 e_1+\cdots+\lambda_r e_r\|_v = \max(|\lambda_1|_v,\ldots,|\lambda_r|_v).$$

We refer readers to [21, § 3] for more details. If $\overline{D} = (D, (g_v)_{v\in M_K})$ is an adelic $\mathbb{R}$-Cartier divisor on X, then

$$\overline{H^0(D)} := (H^0(D), (\|\cdot\|_{g_v})_{v\in M_K})$$

is an adelic vector bundle on $\operatorname{Spec} K$.

A variant of the arithmetic volume function has been introduced by Yuan [39] (see also [34, § 4.3]), where he replaces $\log \#\widehat{H}^0(n\overline{D})$ in the formula (9) by the Euler-Poincaré characteristic of $\overline{H^0(nD)}$:

$$\widehat{\operatorname{vol}}_\chi(\overline{D}) := \limsup_{n\to+\infty} \frac{\chi(\overline{H^0(nD)})}{n^{d+1}/(d+1)!}. \tag{10}$$

This function is called the *χ-volume* function.

2.2.7. Let $\overline{D}$ be an adelic $\mathbb{R}$-Cartier divisor on X such that D is big. Then the family

$$V_\bullet(\overline{D}) := (\overline{H^0(nD)})_{n\in\mathbb{N}}$$

forms an adelically normed graded linear series in the sense of [9]. By using the filtration by height (see [9, § 3.2]), we have constructed a concave and upper semicontinuous function $G_{\overline{D}}$ on $\Delta(D)$, called the *concave transform* of $\overline{D}$, such that

$$\widehat{\operatorname{vol}}(\overline{D}) = (d+1)!\int_{\Delta(D)} \max(G_{\overline{D}}(x), 0)\mathrm{d}x, \tag{11}$$

and

$$\widehat{\operatorname{vol}}_\chi(\overline{D}) = (d+1)!\int_{\Delta(D)} G_{\overline{D}}(x)\mathrm{d}x. \tag{12}$$

This function is positively homogeneous in the following sense: for any $\overline{D} \in \widehat{\operatorname{Big}}_{\mathbb{R}}(X)$ and any $\lambda > 0$, one has

$$\forall\, x\in\Delta(D),\quad G_{\lambda\overline{D}}(\lambda x) = \lambda G_{\overline{D}}(x).$$

If ζ is an arithmetic $\mathbb{R}$-Cartier divisor on $\operatorname{Spec} K$, then one has

$$G_{\overline{D}+\pi^*(\zeta)}(\cdot) = G_{\overline{D}}(\cdot) + \widehat{\deg}(\zeta)$$

on $\Delta(D)$, which implies the following equality

$$\frac{\widehat{\mathrm{vol}}_\chi(\overline{D}+\pi^*(\zeta))}{\mathrm{vol}(D)} = \frac{\widehat{\mathrm{vol}}_\chi(\overline{D})}{\mathrm{vol}(D)} + (d+1)\widehat{\deg}(\zeta). \tag{13}$$

Moreover, if $\overline{D}_1$ and $\overline{D}_2$ are two adelic $\mathbb{R}$-Cartier divisors on X, then for any $(x,y)\in\Delta(D_1)\times\Delta(D_2)$ one has

$$G_{\overline{D}_1+\overline{D}_2}(x+y)\geq G_{\overline{D}_1}(x)+G_{\overline{D}_2}(y).$$

We refer readers to [9, § 2.4] for more details, see also [17, § 3.2 and § 4.2] for the super-additivity of the concave transform.

2.2.8. Let $\overline{D}=(D,(g_v)_{v\in M_K})$ be an adelic $\mathbb{R}$-Cartier divisor on X. We say that $\overline{D}$ is *relatively nef* if the $\mathbb{R}$-Cartier divisor D is nef and if all v-Green functions g_v are plurisubharmonic. In the case where v is non-Archimedean, the plurisubharmonicity of g_v signifies that the Green function g_v is a uniform limit of v-Green functions of D arising from relatively nef integral models. We refer readers to [34, §§ 2.1–2.2, § 4.4] for more details.

The arithmetic intersection number has been defined in [34, § 4.5] for relatively nef adelic $\mathbb{R}$-Cartier divisors. It is a $(d+1)$-linear form on the cone of relatively nef adelic $\mathbb{R}$-Cartier divisors. If $\{\overline{D}_0,\ldots,\overline{D}_d\}$ is a family of relatively nef adelic $\mathbb{R}$-Cartier divisors, we use the expression $\widehat{\deg}(\overline{D}_0\cdots\overline{D}_d)$ to denote the arithmetic intersection number of the adelic $\mathbb{R}$-Cartier divisors $\overline{D}_0,\ldots,\overline{D}_d$.

If $\overline{D}$ is a relatively nef adelic $\mathbb{R}$-Cartier divisor, one can identify the arithmetic self-intersection number $\widehat{\deg}(\overline{D}^{d+1})$ with the χ-volume function. One has

$$\widehat{\mathrm{vol}}_\chi(\overline{D})=\widehat{\deg}(\overline{D}^{d+1}). \tag{14}$$

This follows from the arithmetic Hilbert-Samuel theorem [1, 36] and the continuity of the arithmetic intersection number on the relatively nef cone. In particular, we deduce from (12) that, if $\overline{D}$ is an adelic $\mathbb{R}$-Cartier divisor which is relatively nef, then one has

$$\widehat{\deg}(\overline{D}^{d+1})=(d+1)!\int_{\Delta(D)}G_{\overline{D}}(x)\mathrm{d}x. \tag{15}$$

2.2.9. Given an adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ on X, one can define a height function $h_{\overline{D}}$ on the set of all closed points of X. In particular, when x is a closed point of X which does not lie in the support of D, the height $h_{\overline{D}}(x)$ is the Arakelov degree of the restriction of $\overline{D}$ on x. The adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ is said to be *nef* if it is relatively nef and if the height function $h_{\overline{D}}$ is non-negative (see [34, § 4.4]). If $\overline{D}$ is nef, one has (see [25, Proposition 3.11])

$$\widehat{\deg}(\overline{D}^{(d+1)})=\widehat{\mathrm{vol}}(\overline{D}). \tag{16}$$

The comparison between (11) and (15) shows that, if $\overline{D}$ is nef, then the function $G_{\overline{D}}$ is non-negative almost everywhere on $\Delta(D)$, and hence is non-negative on $\Delta(D)^\circ$, since it is concave.

2.2.10. The arithmetic volume function is differentiable on the cone of big adelic $\mathbb{R}$-Cartier divisors. More precisely, if $\overline{D}$ and $\overline{E}$ are adelic $\mathbb{R}$-Cartier divisors on X, where $\overline{D}$ is big, then the limit

$$\langle \overline{D}^d\rangle \cdot \overline{E} := \lim_{t\to 0} \frac{\widehat{\mathrm{vol}}(\overline{D}+t\overline{E}) - \widehat{\mathrm{vol}}(\overline{D})}{(d+1)t} \tag{17}$$

exists in $\mathbb{R}$, and defines a linear form on $\overline{E} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$. This result was first proved in the case where D and E are Cartier divisors in [15], and then was extended to the general case of adelic $\mathbb{R}$-Cartier divisors in [25] (the normality hypothesis on the arithmetic variety in the differentiability theorem in *loc. cit.* is not necessary, since the arithmetic volume function is invariant by pull-back to a birational modification). We observe from the definition that if $\overline{E}_1$ and $\overline{E}_2$ are adelic $\mathbb{R}$-Cartier divisors such that $\overline{E}_2 - \overline{E}_1$ is effective, then one has

$$\langle \overline{D}^d\rangle \cdot \overline{E}_1 \leq \langle \overline{D}^d\rangle \cdot \overline{E}_2. \tag{18}$$

Moreover, if $\nu : X' \to X$ is a birational projective morphisme, then one has

$$\langle \nu^*(\overline{D})^d\rangle \cdot \nu^*(\overline{E}) = \langle \overline{D}^d\rangle \cdot \overline{E}. \tag{19}$$

This follows from the birational invariance of the arithmetic volume function.

Note that $\langle \overline{D}^d\rangle \cdot \overline{E}$ identifies with the arithmetic positive intersection number introduced in [15]. Recall that, if the adelic $\mathbb{R}$-Cartier divisor $\overline{E}$ is nef, then one has (see [15, § 3.3], see also [25, § 3])

$$\langle \overline{D}^d\rangle \cdot \overline{E} = \sup_{\nu:X'\to X,\ \overline{D}'\leq \nu^*(\overline{D})} \widehat{\deg}(\overline{D}'^d \cdot \nu^*(\overline{E})), \tag{20}$$

where $(\nu, \overline{D}')$ runs over the set of all couples with $\nu : X' \to X$ being a birational modification of X and $\overline{D}'$ is a nef arithmetic $\mathbb{R}$-Cartier divisor on X' such that $\nu^*(\overline{D}) - \overline{D}'$ is effective.

Let $\overline{D}$ be a big $\mathbb{R}$-Cartier divisor on X. We denote by $\mathrm{vol}_+(\overline{D})$ the volume of the graded linear series

$$V^0(\overline{D}) := \bigoplus_{n\geq 0} \mathrm{Vect}_K(\widehat{H}^0(n\overline{D})).$$

This invariant is a birational invariant (its birationality follows from that of the arithmetic volume function and [14, Corollary 4.6]). Moreover, one has

$$\mathrm{vol}_+(\overline{D}) = \sup_{\nu:X'\to X, \overline{D}'\leq \nu^*(\overline{D})} \mathrm{vol}(D'), \tag{21}$$

where $(\nu, \overline{D}')$ runs over the set of all couples with $\nu : X' \to X$ being a birational modification of X and $\overline{D}'$ a nef arithmetic $\mathbb{R}$-Cartier divisor on X' such that $\nu^*(\overline{D}) - \overline{D}'$ is effective. Note that this invariant has also been introduced in [42, § 3.2] as "volume derivative".

3. Relative isoperimetric inequality

The purpose of this section is to establish an integral form of isoperimetric inequality and apply it to the study of the arithmetic volume function. Throughout the section, we fix an integer $d \geq 1$.

3.1. Isoperimetric inequality for integrals. — Let Δ_0 and Δ_1 be two convex bodies in $\mathbb{R}^d$. For any $\varepsilon \in [0,1]$, let S_ε be the Minkowski sum

$$\Delta_0 + \varepsilon\Delta_1 := \{x + \varepsilon y : x \in \Delta_0, y \in \Delta_1\}.$$

It is also a convex body in $\mathbb{R}^d$.

Theorem 3.1. — *Let G_0 and G_1 be two Borel functions on Δ_0 and Δ_1, respectively. We assume that they are integrable with respect to the Lebesgue measure. Suppose given, for any $\varepsilon \in [0,1]$, an almost everywhere non-negative Borel function H_ε on S_ε such that*

$$\forall\,(x,y) \in \Delta_0 \times \Delta_1, \quad H_\varepsilon(x + \varepsilon y) \geq G_0(x) + \varepsilon G_1(y). \tag{22}$$

Then the following inequality holds

$$\begin{aligned}&\liminf_{\varepsilon\to 0+} \frac{\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z - \int_{\Delta_0} G_0(x)\mathrm{d}x}{\varepsilon}\\ &\geq d\left(\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)}\right)^{1/d}\int_{\Delta_0} G_0(x)\mathrm{d}x + \frac{\mathrm{vol}(\Delta_0)}{\mathrm{vol}(\Delta_1)}\int_{\Delta_1} G_1(y)\mathrm{d}y.\end{aligned} \tag{23}$$

Proof. — The key point of the proof is to choose a suitable map $f : \Delta_0 \to \Delta_1$ as an auxiliary tool to relate Δ_0, Δ_1 and S_ε. We consider the Knothe map $f : \Delta_0 \to \Delta_1$ which is a homeomorphism and of class C^1 on Δ_0°, whose Jacobian Df is upper triangular with a positive diagonal everywhere on Δ_0°, and such that $\det(Df)$ is constant (which is necessarily equal to $\mathrm{vol}(\Delta_1)/\mathrm{vol}(\Delta_0)$). We refer readers to [28] and [3, § 2.2.1] for details on the construction of this map. We just point out that we can write the map f in the form

$$f(x_1,\ldots,x_d) = (f_1(x_1), f_2(x_1,x_2),\ldots,f_d(x_1,\ldots,x_d)),$$

where for each $k \in \{1,\ldots,d\}$, f_k is a function from $\mathbb{R}^k$ to $\mathbb{R}$ which is increasing in the variable x_k when other coordinates $(x_1,\ldots,x_{k-1})$ are fixed. Moreover, this monotonicity is strict on the interval of points $x_k \in \mathbb{R}$ such that $(x_1,\ldots,x_k)$ lies in the projection of Δ_0 by taking the first k coordinates (with fixed $(x_1,\ldots,x_{k-1})$ again).

For any $\varepsilon \in [0,1]$, let $F_\varepsilon := \mathrm{Id} + \varepsilon f : \Delta_0 \to S_\varepsilon$ which sends $x \in \Delta_0$ to $x + \varepsilon f(x)$. Note that the map F_ε has the same monotonicity property as f. In particular, the map F_ε is injective on Δ_0°. Therefore by the positivity of the function H_ε one has

$$\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z \geq \int_{F_\varepsilon(\Delta_0)} H_\varepsilon(z)\mathrm{d}z = \int_{\Delta_0^\circ} H_\varepsilon(F_\varepsilon(x))|\det(DF_\varepsilon)(x)|\mathrm{d}x.$$

Note that one has $DF_\varepsilon = \mathrm{Id} + \varepsilon Df$ on Δ_0°. Since Df is upper triangular and the coefficients of its diagonal are positive, one has

$$|\det(DF_\varepsilon)| = \det(\mathrm{Id} + \varepsilon Df) \geq (1 + \varepsilon \det(Df)^{1/d})^d = \left(1 + \varepsilon \left(\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)}\right)^{\frac{1}{d}}\right)^d.$$

Hence we obtain

$$\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z \geq \left(1 + \varepsilon \left(\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)}\right)^{\frac{1}{d}}\right)^d \int_{\Delta_0} H_\varepsilon(F_\varepsilon(x))\mathrm{d}x.$$

By the super-additivity assumption (22), one has

$$H_\varepsilon(F_\varepsilon(x)) = H_\varepsilon(x + \varepsilon f(x)) \geq G_0(x) + \varepsilon G_1(f(x)).$$

Therefore

$$\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z \geq \left(1 + \varepsilon \left(\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)}\right)^{\frac{1}{d}}\right)^d \left(\int_{\Delta_0} G_0(x)\mathrm{d}x + \varepsilon \int_{\Delta_0} G_1(f(x))\mathrm{d}x\right).$$

Since f is a homeomorphism between Δ_0 and Δ_1, and

$$\det(Df) = \mathrm{vol}(\Delta_1)/\mathrm{vol}(\Delta_0)$$

is constant on Δ_0°, one has

$$\int_{\Delta_0} G_1(f(x))\mathrm{d}x = \frac{\mathrm{vol}(\Delta_0)}{\mathrm{vol}(\Delta_1)} \int_{\Delta_1} G_1(y)\mathrm{d}y.$$

Combining with the above inequality, we obtain that

$$\frac{1}{\varepsilon}\left(\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z - \int_{\Delta_0} G_0(x)\mathrm{d}x\right)$$

is bounded from below by

$$\frac{1}{\varepsilon}\Bigg[\left(\mathrm{vol}(\Delta_0)^{\frac{1}{d}} + \varepsilon\,\mathrm{vol}(\Delta_1)^{\frac{1}{d}}\right)^d \left(\frac{\int_{\Delta_0} G_0(x)\mathrm{d}x}{\mathrm{vol}(\Delta_0)} + \varepsilon \frac{\int_{\Delta_1} G_1(y)\mathrm{d}y}{\mathrm{vol}(\Delta_1)}\right) - \int_{\Delta_0} G_0(x)\mathrm{d}x\Bigg].$$

By taking the inf limit when ε tends to $0+$, we obtain the lower bound as stated in the theorem. □

Remark 3.2. — The inequality (23) can be considered as a natural generalization of the classic isoperimetric inequality. In fact, if we take G_0 and G_1 to be the constant function of value 1 on Δ_0 and Δ_1 respectively, and let $H_\varepsilon(z) = 1+\varepsilon$ for any $\varepsilon\in[0,1]$ and any $z\in S_\varepsilon$, then these functions verify the conditions of Theorem 3.1. Moreover, one has

$$\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z = (1+\varepsilon)\operatorname{vol}(S_\varepsilon).$$

Hence

$$\lim_{\varepsilon\to 0+}\frac{\int_{S_\varepsilon} H_\varepsilon(z)\mathrm{d}z - \int_{\Delta_0} G_0(x)\mathrm{d}x}{\varepsilon} = d\operatorname{vol}_{d-1,1}(\Delta_0,\Delta_1)+\operatorname{vol}(\Delta_0),$$

where $\operatorname{vol}_{d-1,1}(\Delta_0,\Delta_1)$ is the mixed volume of index $(d-1,1)$ of Δ_0 and Δ_1. Therefore the inequality (23) leads to

$$\operatorname{vol}_{d-1,1}(\Delta_0,\Delta_1)\geq \operatorname{vol}(\Delta_0)^{(d-1)/d}\cdot\operatorname{vol}(\Delta_1)^{1/d},$$

which is the isoperimetric inequality in convex geometry.

3.2. Relative arithmetic isoperimetric inequality. — Let K be a number field and X be a geometrically integral projective scheme of Krull dimension $d\geq 1$ over $\operatorname{Spec} K$. The purpose of this subsection is to establish the following theorem.

Theorem 3.3. — *Let $\overline{D}_0$ and $\overline{D}_1$ be two adelic $\mathbb{R}$-Cartier divisors on X such that D_0 and D_1 are big. Assume that $\widehat{\operatorname{vol}}(\overline{D}_0)=\widehat{\operatorname{vol}}_\chi(\overline{D}_0)>0$. Then one has*

$$(24)\qquad (d+1)\langle\overline{D}_0^d\rangle\cdot\overline{D}_1\geq d\left(\frac{\operatorname{vol}(D_1)}{\operatorname{vol}(D_0)}\right)^{\frac{1}{d}}\widehat{\operatorname{vol}}_\chi(\overline{D}_0)+\left(\frac{\operatorname{vol}(D_0)}{\operatorname{vol}(D_1)}\right)\widehat{\operatorname{vol}}_\chi(\overline{D}_1).$$

Proof. — For each $\varepsilon\in[0,1]$, let $\overline{E}_\varepsilon$ be the adelic $\mathbb{R}$-Cartier divisor $\overline{D}_0+\varepsilon\overline{D}_1$ and $\Delta(E_\varepsilon)$ be the Newton-Okounkov body of E_ε (see Section 2.1.5). Note that one has

$$\Delta(E_\varepsilon)\supset S_\varepsilon := \Delta(D_0)+\varepsilon\Delta(D_1)$$

where $\Delta(D_0)$ and $\Delta(D_1)$ are, respectively, the Newton-Okounkov bodies of D_0 and D_1. For any $\varepsilon\in[0,1]$, let $G_{\overline{E}_\varepsilon}$ be the concave transform of $\overline{E}_\varepsilon$, which verifies (see Sections 2.2.7–2.2.9)

$$\widehat{\operatorname{vol}}(\overline{E}_\varepsilon)=(d+1)!\int_{\Delta(E_\varepsilon)}\max(G_{\overline{E}_\varepsilon}(z),0)\mathrm{d}z\geq(d+1)!\int_{S_\varepsilon}\max(G_{\overline{E}_\varepsilon}(z),0)\mathrm{d}z.$$

Moreover, for any $x\in\Delta(D_0)$ and any $y\in\Delta(D_1)$ one has

$$\max(G_{\overline{E}_\varepsilon}(x+\varepsilon y),0)\geq G_{\overline{E}_\varepsilon}(x+\varepsilon y)\geq G_{\overline{D}_0}(x)+\varepsilon G_{\overline{D}_1}(y),$$

where $G_{\overline{D}_0}$ and $G_{\overline{D}_1}$ are, respectively, the concave transforms of $\overline{D}_0$ and $\overline{D}_1$. By the assumption $\widehat{\mathrm{vol}}(\overline{D}_0) = \widehat{\mathrm{vol}}_\chi(\overline{D}_0)$ we obtain that (see also (11) and (12))

$$\widehat{\mathrm{vol}}(\overline{D}_0) = (d+1)! \int_{\Delta(D_0)} G_{\overline{D}_0}(x)\mathrm{d}x = \widehat{\mathrm{vol}}_\chi(\overline{D}_0)$$

Therefore, Theorem 3.1 leads to

$$\begin{aligned}(d+1)\langle \overline{D}_0^d\rangle \cdot \overline{D}_1 &= \lim_{\varepsilon\to 0+} \frac{\widehat{\mathrm{vol}}(\overline{E}_\varepsilon) - \widehat{\mathrm{vol}}(\overline{D}_0)}{\varepsilon} \\ &\geq d\left(\frac{\mathrm{vol}(D_1)}{\mathrm{vol}(D_0)}\right)^{1/d} \widehat{\mathrm{vol}}_\chi(\overline{D}_0) + \frac{\mathrm{vol}(D_0)}{\mathrm{vol}(D_1)}\widehat{\mathrm{vol}}_\chi(\overline{D}_1),\end{aligned}$$

as claimed in the theorem. □

Remark 3.4. — The condition $\widehat{\mathrm{vol}}(\overline{D}_0) = \widehat{\mathrm{vol}}_\chi(\overline{D}_0)$ in Theorem 3.3 is fulfilled notably when $\overline{D}_0$ is nef. If in addition $\overline{D}_1$ is relatively nef, then the positive intersection product $\langle \overline{D}_0^d\rangle\cdot\overline{D}_1$ coincides with the arithmetic intersection number $\widehat{\deg}(\overline{D}_0^d\cdot\overline{D}_1)$. Therefore, in the case where $\overline{D}_0$ is nef and big, and $\overline{D}_1$ is relatively nef, the inequality (24) can be rewritten in the form

$$(d+1)\widehat{\deg}(\overline{D}_0^d \cdot \overline{D}_1) \geq d\left(\frac{\mathrm{vol}(D_1)}{\mathrm{vol}(D_0)}\right)^{\frac{1}{d}} \widehat{\deg}(\overline{D}_0^{d+1}) + \left(\frac{\mathrm{vol}(D_0)}{\mathrm{vol}(D_1)}\right)\widehat{\deg}(\overline{D}_1^{d+1}).$$

In the particular case where $d = 1$, we obtain

$$2\widehat{\deg}(\overline{D}_0 \cdot \overline{D}_1) \geq \frac{\deg(D_1)}{\deg(D_0)}\widehat{\deg}(\overline{D}_0^2) + \frac{\mathrm{vol}(D_0)}{\mathrm{vol}(D_1)}\widehat{\deg}(\overline{D}_1^2),$$

which is an equivalent form of the arithmetic Hodge index theorem of Faltings [19] and Hriljac [24]; see [17, Remark 4.15] for the comparison of the above inequality with the original statement. In this sense, Theorem 3.3 can be considered as a higher dimensional generalization of the arithmetic Hodge index theorem. Note that in the literature there exist generalizations of the result of Faltings and Hriljac such as [29, 32, 41]. However, these results are, rather, extensions of the arithmetic Hodge index theorem to the setting of intersection along a two-dimensional arithmetic cycle (in a higher-dimensional arithmetic variety), and hence have a very different nature from that of Theorem 3.3.

Theorem 3.5. — *Let $\overline{D}_0$ and $\overline{D}_1$ be big adelic $\mathbb{R}$-Cartier divisors on X. Then the following inequality holds.*

$$(d+1)\,\langle \overline{D}_0^d\rangle \cdot \overline{D}_1 \geq d\left(\frac{\mathrm{vol}_+(\overline{D}_1)}{\mathrm{vol}_+(\overline{D}_0)}\right)^{\frac{1}{d}} \widehat{\mathrm{vol}}(\overline{D}_0) + \left(\frac{\mathrm{vol}_+(\overline{D}_0)}{\mathrm{vol}_+(\overline{D}_1)}\right)\widehat{\mathrm{vol}}(\overline{D}_1). \tag{25}$$

Proof. — Let $\nu : X' \to X$ be a birational projective morphism and and $\overline{D}_1'$ be a nef and big adelic $\mathbb{R}$-Cartier divisor on X' such that $\nu^*(\overline{D}_1) - \overline{D}_1'$ is effective.

By Remark 3.4, we obtain that, for any birational projective morphism $\pi : X'' \to X'$ and any nef and big adelic $\mathbb{R}$-Cartier divisor $\overline{D}''_0$ on X'' such that $(\nu\pi)^*(\overline{D}_0) - \overline{D}''_0$ is effective, one has

$$(d+1)\widehat{\deg}(\overline{D}''^{d}_0 \cdot \pi^*(\overline{D}'_1)) \geq d\left(\frac{\mathrm{vol}(D'_1)}{\mathrm{vol}(D''_0)}\right)^{\frac{1}{d}} \widehat{\deg}(\overline{D}''^{(d+1)}_0) + \left(\frac{\mathrm{vol}(D''_0)}{\mathrm{vol}(D'_1)}\right) \widehat{\deg}(\overline{D}'^{(d+1)}_1).$$

By the relations (20) and (21), and the arithmetic Fujita approximation property of the arithmetic volume function, we deduce that

$$(d+1)\,\langle \nu^*(\overline{D}_0)^d\rangle \cdot \overline{D}'_1 \geq d\left(\frac{\mathrm{vol}(D'_1)}{\mathrm{vol}_+(\overline{D}_0)}\right)^{\frac{1}{d}} \widehat{\mathrm{vol}}(\overline{D}_0) + \left(\frac{\mathrm{vol}_+(\overline{D}_0)}{\mathrm{vol}(D'_1)}\right) \widehat{\deg}(\overline{D}'^{\,d+1}_1).$$

Moreover, by (18) and (19) one has

$$\langle \overline{D}^d_0\rangle \cdot \overline{D}_1 = \langle \nu^*(\overline{D}_0)^d\rangle \cdot \nu^*(\overline{D}_1) \geq \langle \nu^*(\overline{D}_0)\rangle \cdot \overline{D}'_1.$$

Therefore, still by the relations (20) and (21), and the arithmetic Fujita approximation property of the arithmetic volume function (applied to $\overline{D}_1$), we obtain the inequality (25). □

Remark 3.6. — The inequality between arithmetic and geometric means shows that (25) is a refinement of the arithmetic isoperimetric inequality proved in [15, Proposition 4.5], which asserts that

$$\langle \overline{D}^d_0\rangle \cdot \overline{D}_1 \geq \widehat{\mathrm{vol}}(\overline{D}_0)^{d/(d+1)} \cdot \widehat{\mathrm{vol}}(\overline{D}_1)^{1/(d+1)} \tag{26}$$

under the same notation and hypothesis of Theorem 3.5.

3.3. Equality condition. — It is an interesting problem to determine the conditions under which the equality in (24) holds. We assume that $\overline{D}_1$ is relatively nef and that there exists an arithmetic $\mathbb{R}$-Cartier divisor ζ on $\mathrm{Spec}\,K$ with $\widehat{\deg}(\zeta) > 0$ such that $\overline{D}_0 - \pi^*(\zeta)$ is nef (this condition is satisfied notably when $\overline{D}_0$ is arithmetically ample). If the equality in (24) holds for $\overline{D}_0$ and $\overline{D}_1$, by applying the inequality (24) to $\overline{D}_0 - \pi^*(\zeta)$ and $\overline{D}_1$, we obtain

$$\begin{aligned}&(d+1)\,\widehat{\deg}(\overline{D}^d_0 \cdot \overline{D}_1) - d(d+1)\deg(D^{d-1}_0 \cdot D_1)\widehat{\deg}(\zeta)\\ &\geq d\left(\frac{\mathrm{vol}(D_1)}{\mathrm{vol}(D_0)}\right)^{\frac{1}{d}} \widehat{\deg}(\overline{D}^{d+1}_0) + \frac{\mathrm{vol}(D_0)}{\mathrm{vol}(D_1)}\widehat{\deg}(\overline{D}^{d+1}_1)\\ &- d(d+1)\left(\frac{\mathrm{vol}(D_1)}{\mathrm{vol}(D_0)}\right)^{\frac{1}{d}} \widehat{\deg}(\zeta),\end{aligned}$$

which leads to

$$\deg(D^{d-1}_0 \cdot D_1) \leq \mathrm{vol}(D_1)^{1/d}\,\mathrm{vol}(D_0)^{(d-1)/d}. \tag{27}$$

By the isoperimetric inequality in algebraic geometry (which is a direct consequence of the Brunn-Minkowski inequality), the reverse inequality of (27) holds, which leads to

$$\deg(D_0^{d-1}\cdot D_1)=\operatorname{vol}(D_1)^{1/d}\operatorname{vol}(D_0)^{(d-1)/d}.$$

By [10, Theorem D], we obtain that the classes of D_0 and D_1 in the Néron-Severi space of X are collinear. Note that the equality in (24) is stable by dilations of $\overline{D}_0$ and $\overline{D}_1$ by positive numbers. Hence we may assume without loss of generality that $\operatorname{vol}(D_0)=\operatorname{vol}(D_1)$. In this case the equality in (24) becomes

$$(d+1)\widehat{\deg}(\overline{D}_0^{d}\cdot\overline{D}_1)=d\widehat{\deg}(\overline{D}_0^{d+1})+\widehat{\deg}(\overline{D}_1^{d+1}). \tag{28}$$

This equality remains true if we replace $\overline{D}_0$ (resp. $\overline{D}_1$) by $\overline{D}_0+\pi^*(\zeta_0)$ and $\overline{D}_0+\pi^*(\zeta_1)$, where ζ_0 and ζ_1 are adelic $\mathbb{R}$-Cartier divisors on $\operatorname{Spec}K$. Hence we may assume without loss of generality that $\overline{D}_0$ and $\overline{D}_1$ are nef and big and one has $\widehat{\deg}(\overline{D}_0^{d+1})=\widehat{\deg}(\overline{D}_1^{d+1})$. Then by [25, Theorem 7.4], we obtain that $\overline{D}_1-\overline{D}_0$ is a principal adelic $\mathbb{R}$-Cartier divisor.

We re-cast the above observation into a proposition as follows.

Proposition 3.7. — *Assume that $\overline{D}_1$ is relatively nef and that there exists an arithmetic $\mathbb{R}$-Cartier divisor ζ on $\operatorname{Spec}K$ with $\widehat{\deg}(\zeta)>0$ such that $\overline{D}_0-\pi^*(\zeta)$ is nef, the equality in (24) holds if and only if there exists an element $\phi\in K(X)^{\times}_{\mathbb{R}}$ and an adelic $\mathbb{R}$-Cartier divisor ξ on $\operatorname{Spec}K$ such that*

$$\frac{\overline{D}_1}{\operatorname{vol}(D_1)^{1/d}}=\frac{\overline{D}_0}{\operatorname{vol}(D_0)^{1/d}}+\widehat{\operatorname{div}}(\phi)+\pi^*(\xi).$$

Remark 3.8. — The inequality (25) was deduced from (24) (for a family of adelic $\mathbb{R}$-Cartier divisor pairs), however the comparison of the terms on the right-hand sides of these inequalities remains obscure, even in the case where $\overline{D}_0$ is nef (and hence $\widehat{\operatorname{vol}}_\chi(\overline{D}_0)=\widehat{\operatorname{vol}}(\overline{D}_0)$ and $\operatorname{vol}_+(\overline{D}_0)=\operatorname{vol}(D_0)$). Certainly $\widehat{\operatorname{vol}}_\chi(\overline{D}_1)/\operatorname{vol}(D_1)$ is bounded from above by $\widehat{\operatorname{vol}}(\overline{D}_1)/\operatorname{vol}_+(\overline{D}_1)$ because the former is the average of the function $G_{\overline{D}_1}$, while the latter is the average of the positive part of the same function. However, $\operatorname{vol}_+(\overline{D}_1)$ is bounded from above by $\operatorname{vol}(D_1)$. It seems to me a more subtle problem to determine the equality condition of the inequality (25).

3.4. Function field case. — The comparison between the inequalities (24) and (26) shows that Theorem 3.3 can be considered as a refinement of the isoperimetric inequality where we take into account the information of X relatively to the arithmetic curve $\operatorname{Spec}K$. The same method can also be applied to the function field setting, which leads to the following relative form of the isoperimetric inequality in algebraic geometry. We refer readers to [16, § 8] for the construction of the concave transform in the function field setting. In order to be more explicative, we avoid introducing adelic divisors on the function field setting by only asserting the statement for nef and big line bundles. Readers may keep in mind that the result still holds for general adelic $\mathbb{R}$-Cartier divisors with the same proof as in the arithmetic case.

THEOREM 3.9. — *Let k be a field, C be a regular projective curve over* $\operatorname{Spec} k$, *and $\pi : X \to C$ be a flat and projective k-morphism of relative dimension $d \geq 1$. If L and M are two nef and big line bundles on X, then one has*

$$(d+1)(c_1(L)^d \cdot c_1(M)) \geq d\left(\frac{c_1(M_\eta)^d}{c_1(L_\eta)^d}\right)^{1/d} c_1(L)^{d+1} + \left(\frac{c_1(L_\eta)^d}{c_1(M_\eta)^d}\right) c_1(M)^{d+1},$$

where η is the generic point of C, and L_η and M_η are, respectively, the restrictions of L and M on the generic fiber of π.

4. Relative Brunn-Minkowski inequality

The purpose of this section is to establish the following relative form of the Brunn-Minkowski inequality in the arithmetic geometry setting.

THEOREM 4.1. — *Let K be a number field and X be a geometrically integral projective scheme over* $\operatorname{Spec} K$. *If $\overline{D}_1, \ldots, \overline{D}_n$ are adelic $\mathbb{R}$-Cartier divisors on X such that $D_1, \ldots, D_n$ are big and that*

$$\widehat{\mathrm{vol}}(\overline{D}_1 + \cdots + \overline{D}_n) = \widehat{\mathrm{vol}}_\chi(\overline{D}_1 + \cdots + \overline{D}_n) > 0,$$

then one has

$$\frac{\widehat{\mathrm{vol}}(\overline{D}_1 + \cdots + \overline{D}_n)}{\mathrm{vol}(D_1 + \cdots + D_n)} \geq \varphi(D_1, \ldots, D_n)^{-1} \sum_{i=1}^{n} \frac{\widehat{\mathrm{vol}}_\chi(\overline{D}_i)}{\mathrm{vol}(D_i)}, \tag{29}$$

where

$$\varphi(D_1, \ldots, D_n) := d + 1 - d\frac{\mathrm{vol}(D_1)^{1/d} + \cdots + \mathrm{vol}(D_n)^{1/d}}{\mathrm{vol}(D_1 + \cdots + D_n)^{1/d}}. \tag{30}$$

Proof. — Let $\overline{D} = \overline{D}_1 + \cdots + \overline{D}_n$. This is a big adelic $\mathbb{R}$-Cartier divisor. By definition, one has

$$\widehat{\mathrm{vol}}(\overline{D}) = \langle \overline{D}^d \rangle \cdot \overline{D} = \sum_{i=1}^{n} \langle \overline{D}^d \rangle \cdot \overline{D}_i.$$

By Theorem 3.3, one has

$$(d+1)\langle \overline{D}^d \rangle \cdot \overline{D}_i \geq d\left(\frac{\mathrm{vol}(D_i)}{\mathrm{vol}(D)}\right)^{1/d} \widehat{\mathrm{vol}}(\overline{D}) + \left(\frac{\mathrm{vol}(D)}{\mathrm{vol}(D_i)}\right) \widehat{\mathrm{vol}}_\chi(\overline{D}_i). \tag{31}$$

Therefore we obtain

$$\begin{aligned}(d+1)\widehat{\mathrm{vol}}(\overline{D}_1+\cdots+\overline{D}_n)\\ \geq d\frac{\mathrm{vol}(D_1)^{1/d}+\cdots+\mathrm{vol}(D_n)^{1/d}}{\mathrm{vol}(D_1+\cdots+D_n)^{1/d}}\widehat{\mathrm{vol}}(\overline{D}_1+\cdots+\overline{D}_n)\\ +\mathrm{vol}(D_1+\cdots+D_n)\sum_{i=1}^n\frac{\widehat{\mathrm{vol}}_\chi(\overline{D}_i)}{\mathrm{vol}(D_i)},\end{aligned}$$

which leads to (29). □

Remark 4.2. — Assume that the adelic $\mathbb{R}$-Cartier divisors $\overline{D}_1,\ldots,\overline{D}_n$ are nef and big, and there exists an adelic $\mathbb{R}$-Cartier divisor ζ on $\operatorname{Spec}K$ with $\widehat{\deg}(\zeta)>0$ such that $\overline{D}_1+\cdots+\overline{D}_n-\pi^*(\zeta)$ is nef, where $\pi:X\to\operatorname{Spec}K$ denotes the structural morphism. If the equality in (29) holds, then for any $i\in\{1,\ldots,n\}$, the equality in (31) also holds. By Proposition 3.7 we obtain that there exists adelic $\mathbb{R}$-Cartier divisors $\xi_1,\ldots,\xi_n$ on $\operatorname{Spec}K$ and $\phi_1,\ldots,\phi_n$ in $K(X)_{\mathbb{R}}^{\times}$ such that

$$\frac{\overline{D}_1}{\mathrm{vol}(D_1)^{1/d}}-\widehat{\mathrm{div}}(\phi_1)-\pi^*(\xi_1)=\cdots=\frac{\overline{D}_n}{\mathrm{vol}(D_n)^{1/d}}-\widehat{\mathrm{div}}(\phi_n)-\pi^*(\xi_n).$$

It is not hard to see that the converse is also true: if the above equalities hold, then so, also does the equality in (29).

By using the same argument, we deduce from Theorem 3.9 the following relative Brunn-Minkowski inequality in the algebraic geometry setting.

Theorem 4.3. — *Let k be a field, C be a regular projective curve over* $\operatorname{Spec}k$, *and $\pi:X\to C$ be a flat and projective k-morphism of relative dimension $d\geq 1$. If $L_1,\ldots,L_n$ is a family of nef and big line bundles on X, then one has*

$$\frac{\mathrm{vol}(L_1\otimes\cdots\otimes L_n)}{\mathrm{vol}(L_{1,\eta}\otimes\cdots\otimes L_{n,\eta})}\geq\varphi(L_{1,\eta},\ldots,L_{n,\eta})^{-1}\sum_{i=1}^n\frac{\mathrm{vol}(L_i)}{\mathrm{vol}(L_{i,\eta})},$$

where η is the generic point of C, $L_{i,\eta}$ is the restrictions of L_i on the generic fiber of π, and

$$\varphi(L_{1,\eta},\ldots,L_{n,\eta}):=d+1-d\frac{\mathrm{vol}(L_{1,\eta})^{1/d}+\cdots+\mathrm{vol}(L_{n,\eta})^{1/d}}{\mathrm{vol}(L_{1,\eta}\otimes\cdots\otimes L_{n,\eta})^{1/d}}.$$

Remark 4.4. — The infinitesimal argument in Theorem 3.3 is a key step for the inequality (29). In fact, if we apply directly the map of Knothe, as in the proof of Theorem 3.1 with $\varepsilon=1$, we obtain that, for nef adelic $\mathbb{R}$-Cartier divisors $\overline{D}_1$ and $\overline{D}_2$ such that D_1 and D_2 are big, one has

$$\widehat{\mathrm{vol}}(\overline{D}_1+\overline{D}_2)\geq\left(1+\left(\frac{\mathrm{vol}(D_2)}{\mathrm{vol}(D_1)}\right)^{1/d}\right)^d\left(\widehat{\mathrm{vol}}(\overline{D}_1)+\frac{\mathrm{vol}(D_1)}{\mathrm{vol}(D_2)}\widehat{\mathrm{vol}}(\overline{D}_2)\right),$$

which leads to

$$\frac{\widehat{\mathrm{vol}}(\overline{D}_1+\overline{D}_2)}{\mathrm{vol}(D_1+D_2)} \geq \frac{(\mathrm{vol}(D_1)^{1/d}+\mathrm{vol}(D_2)^{1/d})^d}{\mathrm{vol}(D_1+D_2)}\left(\frac{\widehat{\mathrm{vol}}(\overline{D}_1)}{\mathrm{vol}(D_1)}+\frac{\widehat{\mathrm{vol}}(\overline{D}_2)}{\mathrm{vol}(D_2)}\right).$$

However, one has

$$\varphi(D_1,D_2) \leq \frac{\mathrm{vol}(D_1+D_2)}{(\mathrm{vol}(D_1)^{1/d}+\mathrm{vol}(D_2)^{1/d})^d},$$

and the inequality is, in general, strict.

Similarly to Theorem 4.1, we deduce from Theorem 3.5 the following result, which is a refinement of the arithmetic Brunn-Minkowski inequality.

THEOREM 4.5. — *Let K be a number field and X be a geometrically integral scheme over* $\operatorname{Spec} K$*. If $\overline{D}_1,\ldots,\overline{D}_n$ are big adelic $\mathbb{R}$-Cartier divisors on X, then the following inequality holds:*

$$\frac{\widehat{\mathrm{vol}}(\overline{D}_1+\cdots+\overline{D}_n)}{\mathrm{vol}_+(\overline{D}_1+\cdots+\overline{D}_n)} \geq \widehat{\varphi}(\overline{D}_1+\cdots+\overline{D}_n)^{-1}\sum_{i=1}^n \frac{\widehat{\mathrm{vol}}(\overline{D}_i)}{\mathrm{vol}_+(\overline{D}_i)}, \tag{32}$$

where

$$\widehat{\varphi}(\overline{D}_1,\ldots,\overline{D}_n) = d+1-d\frac{\mathrm{vol}_+(\overline{D}_1)^{1/d}+\cdots+\mathrm{vol}_+(\overline{D}_n)^{1/d}}{\mathrm{vol}_+(\overline{D}_1+\cdots+\overline{D}_n)^{1/d}}.$$

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