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RESEARCH ARTICLE

On subfiniteness of graded linear series

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Abstract
Hilbert's fourteenth problem studies the finite generation property of the intersection of an integral algebra of finite type with a subfield of the fraction field of the algebra. It has a negative answer due to a counterexample of Nagata. We show that a subfinite version of Hilbert's fourteenth problem has an affirmative answer. We then establish a graded analogue of this result, which permits to show that the subfiniteness of graded linear series does not depend on the function field in which we consider it. Finally, we apply the subfiniteness result to the study of geometric and arithmetic graded linear series.

Keywords Hilberts fourteenth problem · Algebra of subfinite type · Graded linear series · Newton–Okounkov bodies

Mathematics Subject Classification 14G40 · 11G30

1 Introduction

Let k be a field and X be an integral projective scheme over $\operatorname{Spec} k$. If D is a Cartier divisor on X, as a *graded linear series* of D, one refers to a graded sub-k-algebra of $\bigoplus_{n\in\mathbb{N}} H^0(X, nD)$. The graded linear series are closely related to the positivity of the divisor and are objects of central interest in the study of the geometry of the

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underlying polarised scheme (X, D). Classically the asymptotic behaviour of graded linear series of finite type is well understood through the theory of Hilbert polynomials. Several results in birational algebraic geometry, such as Fujita's approximation theorem [8,28], show that certain graded linear series, even though not of finite type, still have a similar asymptotic behaviour as in the finite generation case. More recently, Lazarsfeld–Mustaţă [15] and Kaveh–Khovanskii [12,13] have proposed, after ideas of Okounkov [24,25], a method to encode the asymptotic behaviour of dimensions of the homogeneous components of a given graded linear series into a convex body (called the *Newton–Okounkov body*) in a Euclidean space.

Note that a graded linear series of a Cartier divisor is always a graded subalgebra of a graded algebra of finite type. It is then quite natural to ask if there is a nice birational geometry for algebras of subfinite type (namely subalgebras of an algebra of finite type) over a field.

From the point of view of birational geometry, it is more convenient to consider graded linear series of a finitely generated field extension K/k without specifying a polarised model of K. In this framework, as a *graded linear series* of K/k, we refer to a graded sub-k-algebra $V_\bullet$ of the polynomial algebra $K[T]$ such that $V_0 = k$ and that V_n is a finite-dimensional vector space over k for any $n \in \mathbb{N}$. In [5], a new construction of Newton–Okounkov bodies has been proposed by using ideas from Arakelov geometry, which only depends on a choice of a tower of successive field extensions $k = K_0 \subset K_1 \subset \cdots \subset K_d = K$ such that each extension K_{i+1}/K_i is transcendental and of transcendence degree 1. The construction is valid for graded linear series $V_\bullet$ of subfinite type (namely contained in a graded linear series of finite type of K/k) whose field of rational functions $k(V_\bullet)$ coincides with K (see Definition 3.1). More precisely, the graded linear series $V_\bullet$ determines, for any $i \in \{0, \ldots, d\}$, a graded linear series $V_{\bullet,K_i}$ of K/K_i by extension of scalars. Moreover, if we denote by C_i the regular projective curve over Spec K_{i-1} whose field of rational functions coincides with K_i, then $V_{\bullet,K_{i-1}}$ generates a graded algebra $E^{(i)}_\bullet$ of vector bundles on C_i, whose generic fibre coincides with $V_{\bullet,K_i}$. We then construct by induction a sequence of convex bodies $\Delta^{(i)}(V_\bullet) \subset \mathbb{R}^{d-i}$, $i \in \{0, \ldots, d\}$, such that $\Delta^{(i-1)}(V_\bullet)$ is the graph of the concave transform of the filtration by minima on $V_{\bullet,K_i}$ associated with the graded algebra of vector bundles $E^{(i)}_\bullet$ (the concave transform here is a concave function defined on the convex body $\Delta^{(i)}(V_\bullet)$). We refer the readers to [1, Section 2.4] for the construction of concave transform of the filtration by minima in the number field setting and to [4, Section 8] for its function field analogue. Our alternative version of Newton–Okounkov body is given by the convex body $\Delta^{(0)}(V_\bullet)$. We emphasise that this construction is quite different compared to the classic one of Kaveh and Khovanskii which arises from a $\mathbb{Z}^d$-valuation with one-dimensional leaves on the field K over k (see [13, Section 2.1] for this notion). See [3] for some explicit computations in the case of a projective bundle over a curve.

The above alternative construction of Newton–Okounkov body is particularly interesting when the extension K/k is not geometrically integral because in this case there does not exist a $\mathbb{Z}^d$-valuation with one-dimensional leaves (see Remark 6.3 for more details). One may expect that the method applies to general graded linear series of subfinite type $V_\bullet$ by considering $V_\bullet$ as a graded linear series of $k(V_\bullet)/k$. However, the main obstruction to this strategy is that *a priori* the condition of subfiniteness depends on the extension K/k with respect to which we consider the graded linear series. This leads to the following subfiniteness problem: given a graded linear series $V_\bullet$ of K/k of subfinite type, does there exist a graded linear series $W_\bullet$ of finite type of the extension $k(V_\bullet)/k$ which contains $V_\bullet$?

Note that the above problem is closely related to Hilbert's fourteenth problem[1]. In fact, given a graded linear series $V_\bullet$ of K/k which is contained in a graded linear series of finite type $V'_\bullet$. The intersection of $V'_\bullet$ with $k(V_\bullet)[T]$ gives a graded linear series of $k(V_\bullet)/k$ containing $V_\bullet$, where $k(V_\bullet)$ is the field of rational functions of $V_\bullet$. Unfortunately the intersection is not necessarily a k-algebra of finite type, as is shown by Nagata's counterexamples [22,23] to Hilbert's fourteenth problem.

Note that the above subfiniteness problem actually asks for a weaker condition than the finite generation of the intersection of $V'_\bullet$ with $k(V_\bullet)[T]$. It suffices that the intersection is contained in a graded linear series of finite type of $k(V_\bullet)$. Similarly, we can consider the following subfinite version of Hilbert's fourteenth problem, which actually has a positive answer (see Theorem 2.6 and Corollary 2.7 *infra*).

Theorem 1.1 *Let k be a field, R be an integral k-algebra of finite type and K be the fraction field of R. Let K' be an extension of k which is contained in K. Then there exists a finitely generated sub-k-algebra R' of K' containing $R \cap K'$, such that* $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$.

The method of proof consists of an induction argument with respect to the field extension K/k which permits to reduce the problem to the case where the extension K/k is monogenerated. Similar method can be applied to the graded case (but with more subtleties because of the grading structure), which leads to the following result and gives an affirmative answer to the subfiniteness problem of graded linear series. It shows that the subfiniteness of graded linear series is an absolute condition, which does not depend on the choice of field extension with respect to which the graded linear series is considered (see Theorem 3.7 and Corollary 4.9 *infra*).

Theorem 1.2 *Let k be a field and K/k be a finitely generated field extension. Let $V_\bullet$ be a graded linear series of K/k which is of subfinite type. Then there exists a graded linear series of finite type $W_\bullet$ of K/k such that $V_\bullet \subset W_\bullet$ and $k(V_\bullet) = k(W_\bullet)$.*

Recall that Hilbert's fourteenth problem is reformulated in a geometric setting by Zariski [29], see also [21] and the survey article [20]. Note that Theorem 1.1 can be compared with the following result in [29, p. 157].

Theorem 1.3 (Zariski) *Let k be a field, A an integrally closed k-algebra of finite type, $K := \mathrm{Frac}(A)$, and K'/k a subextension of K/k. There then exist an integrally closed k-algebra B of finite type and an ideal I of B such that the fraction field of B is k-isomorphic to the fraction field of $A \cap K'$ and that*

$$A \cap K' = \bigcup_{n \in \mathbb{N}} (B : I^n),$$

[1] Let k be a field and $k(x_1, \ldots, x_n)$ be the field of rational functions of n variables. Hilbert's fourteenth problem asked whether the intersection of a subfield of $k(x_1, \ldots, x_n)$ and the polynomial algebra $k[x_1, \ldots, x_n]$ is finitely generated over k (as a k-algebra).

where $(B:I^n) := \{x \in \mathrm{Frac}(B) : xI^n \subset B\}$ *denotes the ideal quotient.*

Inspired by this result, we establish the following projective version of Zariski's theorem and deduce an alternative proof for Theorem 1.2 (see Corollary 4.9 *infra*).

Theorem 1.4 *Let* $K/K'/k$ *be field extensions of finite type and* $W_\bullet$ *a graded linear series of* K/k *that is generated over* k *by the homogeneous elements of degree* 1. *We assume that* W_1 *contains* $1 \in K$ *and that the projective spectrum* $P := \mathrm{Proj}(W_\bullet)$ *is a normal scheme. Let* X *be any integral normal projective* k*-scheme whose field of rational functions is* k*-isomorphic to* $k(W_\bullet \cap K'[T])$. *Then there exists a* $\mathbb{Q}$*-Weil divisor* D *on* X *such that*

$$W_n \cap K' \subset H^0(X, nD) \subset k(W_\bullet \cap K'[T])$$

for every sufficiently positive n.

As an application of the above subfiniteness results, we establish a Fujita approximation theorem for general graded linear series of subfinite type (see Theorem 6.2 *infra*) and an upper bound for the Hilbert–Samuel function of such graded linear series (see Theorem 6.4 *infra*). More precisely, we obtain the following results.

Theorem 1.5 *Let* K/k *be a finitely generated field extension. For any graded linear series* $V_\bullet$ *of* K/k *of subfinite type, whose Kodaira–Iitaka dimension* d *is nonnegative, the limit*

$$\mathrm{vol}(V_\bullet) = \lim_{\substack{n\in\mathbb{N}\\ V_n\neq\{0\}\\ n\to+\infty}} \frac{\dim_k(V_n)}{n^d/d!}$$

exists in $(0,+\infty)$. *Moreover,* $\mathrm{vol}(V_\bullet)$ *is equal to the supremum of* $\mathrm{vol}(W_\bullet)$*, where* $W_\bullet$ *runs over the set of all graded linear series of finite type contained in* $V_\bullet$ *having* d *as the Kodaira–Iitaka dimension. Finally, there exists a function* $f : \mathbb{N} \to \mathbb{R}_+$ *such that*

$$f(n) = \mathrm{vol}(V_\bullet)\,\frac{n^d}{d!} + \mathrm{O}(n^{d-1})$$

and that $\dim_k(V_n) \leqslant f(n)$ *for any* $n \in \mathbb{N}$.

In the case where K admits a $\mathbb{Z}^d$-valuation over k with one-dimensional leaves, we recover a previous result of Kaveh and Khovanskii [13, Corollary 3.11 (2)]. We also apply the above results to the study of graded linear series in the arithmetic setting (see Theorem 6.7 *infra*).

The article is organised as follows. In Sect. 2, we prove a weaker form of Hilbert's fourteenth problem; namely the subfiniteness result stated in Theorem 1.1. In Sect. 3, we prove a graded analogue of Theorem 1.1 in the setting of graded linear series. In Sect. 4 we consider the subfiniteness problem in the geometric setting as a projective analogue of Zariski's result and establish Theorem 1.4. Finally Sect. 5, we develop various applications.

Notation and conventions

1. The field of fractions of an integral domain A is denoted by $\mathrm{Frac}(A)$.
2. Let K/k be an extension of fields. We denote by $\mathrm{tr.deg}_k(K)$ the transcendence degree of K over k.
3. Let S be a scheme. For any $i \in \mathbb{N}$, we denote by $S^{(i)}$ the set of points x of S such that the local ring $\mathcal{O}_{S,x}$ has i as its Krull dimension. If S is an integral scheme, we denote by $\mathrm{Rat}(S)$ the field of rational functions on S.
4. Let k be a field and X be a projective normal scheme over $\mathrm{Spec}\, k$. By a *Weil divisor* (resp. $\mathbb{Q}$-*Weil divisor*) on X, one means an element
$$D = \sum_{V \in X^{(1)}} n_V V$$
in $\mathbb{Z}^{\oplus X^{(1)}}$ (resp. $\mathbb{Q}^{\oplus X^{(1)}}$). The coefficient n_V is referred to as the *multiplicity of* D *along* V, and is denoted by $\mathrm{mult}_V(D)$. If all coefficients n_V are nonnegative, we say that D is *effective*, denoted by $D \geqslant 0$. If ϕ is a nonzero rational function on X, we denote by (ϕ) the principal Weil divisor associated with ϕ, namely
$$(\phi) := \sum_{V \in X^{(1)}} \mathrm{ord}_V(\phi) V.$$
The map $(\cdot)\colon \mathrm{Rat}(X)^\times \to \mathbb{Z}^{\oplus X^{(1)}}$ is a group homomorphism and induces a $\mathbb{Q}$-linear map from $\mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}^{\oplus X^{(1)}}$ which we denote by $(\cdot)_{\mathbb{Q}}$. If D is a $\mathbb{Q}$-Weil divisor on S, we define
$$H^0(X, D) := \left\{\phi \in \mathrm{Rat}(S)^\times : D + (\phi \otimes 1)_{\mathbb{Q}} \geqslant 0\right\} \cup \{0\}$$
and
$$R(D)_\bullet := \bigoplus_{n \geqslant 0} H^0(X, nD) T^n.$$
Note that $R(D)_\bullet$ is a graded sub-k-algebra of the polynomial algebra $\mathrm{Rat}(X)[T]$.
5. Let K be a field. A *discrete valuation* of K means a valuation $\nu\colon K \to \mathbb{Q} \cup \{+\infty\}$ such that $\nu(K^\times)$ is a discrete (or, equivalently, cyclic) subgroup of $(\mathbb{Q}, +)$ (in particular, $\nu(a) = +\infty$ if and only if $a = 0$). Given such a valuation ν, we denote by $O_\nu := \{f \in K : \nu(f) \geqslant 0\}$ its valuation ring, $\mathfrak{m}_\nu$ the maximal ideal of O_ν and $\kappa(\nu) := O_\nu/\mathfrak{m}_\nu$ the residue field. If O_ν is equal to K, we say that the valuation ν is *trivial* (note that in this case $\nu(a) = 0$ for any $a \in K^\times$).
If K/k is a field extension, a *discrete valuation of* K *over* k means a discrete valuation ν of K such that $\nu(a) = 0$ for any $a \in k^\times$. In this case $\kappa(\nu)$ is an extension of k and O_ν is a k-algebra. Two discrete valuations ν_1 and ν_2 of K over k are said to be *equivalent* if there exists an order-preserving isomorphism $\iota\colon \nu_1(K^\times) \to \nu_2(K^\times)$ such that $\nu_2 = \iota \circ \nu_1$.
Let K'/k be a subextension of K/k and let ν be a discrete valuation of K over k which is nontrivial. Then the restriction of ν to K' is a discrete valuation of K' over

k. We define the *ramification index* of ν with respect to K' as the unique integer $e(K',\nu) \in \mathbb{N}$ satisfying

$$\nu(K'^{\times}) = e(K', \nu)\,\nu(K^{\times}).$$

Note that $e(K', \nu) = 0$ if and only if $\nu|_{K'}$ is trivial.

6. Let k be a field and S be an integral separated k-scheme. Given a discrete valuation ν of $\mathrm{Rat}(S)$ over k, we say that a point x of S is the *centre* of ν in S if

$$\mathcal{O}_{S,x} \subset O_\nu \quad \text{and} \quad \mathfrak{m}_x = \mathfrak{m}_\nu \cap \mathcal{O}_{S,x},$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{S,x}$. By the valuative criterion of separation, if the centre of ν in S exists, then it is unique. In the case where the centre of ν in S exists, we denote it by $c_S(\nu)$. If S is proper over k, then by the valuative criterion of properness every discrete valuation of $\mathrm{Rat}(S)$ over k has a centre in S. A discrete valuation ν is trivial if and only if the centre of ν in S is the generic point. Moreover, each regular point $\xi \in S^{(1)} \cup S^{(0)}$ defines a discrete valuation $\mathrm{ord}_\xi : \mathrm{Rat}(S) \to \mathbb{Z} \cup \{+\infty\}$ whose centre is ξ (see Item 3. for the notation of $S^{(0)}$ and $S^{(1)}$).

7. Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded ring. We denote by $\mathrm{Proj}(R_\bullet)$ the projective spectrum of $R_\bullet$. If $M_\bullet$ is a graded $R_\bullet$-module, we denote by $\widetilde{M_\bullet}$ the quasi-coherent $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}$-module associated with $M_\bullet$ (see [9, Section II.2.5]). For any $m \in \mathbb{N}$, we let $M(m)_\bullet$ be the $\mathbb{N}$-graded $R_\bullet$-module such that $M(m)_n = M_{n+m}$ for any $n \in \mathbb{N}$, and let $M_{\geqslant m}$ be the $\mathbb{N}$-graded sub-$R_\bullet$-module of $M_\bullet$ such that $(M_{\geqslant m})_n = \{0\}$ if $n < m$ and $(M_{\geqslant m})_n = M_n$ if $n \geqslant m$. In particular, one has $M(m)_\bullet = M_{\geqslant m}(m)_\bullet$. The quasi-coherent sheaf $\widetilde{R(m)_\bullet}$ is denoted by $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m)$. Note that if $R_\bullet$ is generated as an R_0-algebra by R_1, then $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m)$ are invertible $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}$-modules for all $m \in \mathbb{N}$, and one has canonical isomorphisms

$$\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m) \otimes_{\mathcal{O}_{\mathrm{Proj}(R_\bullet)}} \mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m') \cong \mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m+m')$$

for all $m, m' \in \mathbb{N}$.

8. Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded ring. We say that $R_\bullet$ is *essentially integral* if the ideal $R_{\geqslant 1}$ of $R_\bullet$ is not equal to zero and if the product of two nonzero homogeneous elements of positive degree of $R_\bullet$ is nonzero. Note that if $R_\bullet$ is essentially integral then the scheme $\mathrm{Proj}(R_\bullet)$ is integral (see [9, Proposition II.2.4.4]).

2 A weak form of Hilbert's fourteenth problem

Let k be a field, R be a finitely generated integral k-algebra and K be the field of fractions of R. Clearly K is a finitely generated extension of k. Let K' be a subextension of K/k, which is necessarily a finitely generated extension (see [2, Chapter V, Section 14, n°7, Corollary 3]). We consider the intersection $R \cap K'$ and ask the following question which could be considered as a weaker form of Hilbert's fourteenth problem: *does there exist a finitely generated sub-k-algebra R' of K' containing $R \cap K'$ such* *that* $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$. In this section, we give an affirmative answer to this question.

Definition 2.1 Let k be a field and A be a k-algebra. We say that A is of *subfinite type* if it is a sub-k-algebra of a k-algebra of finite type.

Lemma 2.2 *An injective homomorphism of rings* $A \to B$ *yields a dominant morphism* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$.

Proof Let $\mathfrak{p}$ be a minimal prime ideal of A and $S := A \setminus \mathfrak{p}$. Since the homomorphism of rings $A \to B$ is injective, so is the localised homomorphism $A_{\mathfrak{p}} \to S^{-1}B$. Hence $S^{-1}B$ is nonzero. In particular, there exists a prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P} \cap S = \varnothing$, or equivalently, $\mathfrak{P} \cap A \subset \mathfrak{p}$. Since $\mathfrak{P} \cap A$ is a prime ideal of A and $\mathfrak{p}$ is a minimal prime ideal of A, one has $\mathfrak{P} \cap A = \mathfrak{p}$. □

Proposition 2.3 *Let k be a field and A be a k-algebra of subfinite type. We assume that A is an integral domain. Then there exists a k-algebra of finite type containing A, which is also an integral domain.*

Proof Let B be a k-algebra of finite type such that $A \subset B$. By Lemma 2.2, one can find a prime ideal $\mathfrak{p}$ of B such that $\mathfrak{p} \cap A = \{0\}$. Therefore we can consider A as a sub-k-algebra of $B/\mathfrak{p}$. Since B is a k-algebra of finite type, also is $B/\mathfrak{p}$. □

Lemma 2.4 *Let A be a k-algebra which is an integral domain, and K the field of fractions of A. Let K'/K be a finite extension of K generated by one element α and B' a sub-k-algebra of finite type of K' which contains A. Then there exists a sub-k-algebra of finite type B of K which contains A.*

Proof Let $f \in K[T]$ be the minimal polynomial of α over K, which we assume to be monic. Let $F_1, \ldots, F_n$ be polynomials in $K[T]$ such that $B' = k[F_1(\alpha), \ldots, F_n(\alpha)]$. Let $S \subset K$ be the (finite) set of the coefficients of the polynomials $f, F_1, \ldots, F_n$. We claim that A is contained in $k[S]$. In fact, suppose that an element u of A is written in the form $\varphi(F_1(\alpha), \ldots, F_n(\alpha))$, where $\varphi \in k[X_1, \ldots, X_n]$, then by Euclidean division the polynomial $\varphi(F_1, \ldots, F_n) \in k[S][T]$ can be written as $fg + h$, where g and h are polynomials in $k[S][T]$ with $\deg(h) < \deg(f)$. The decomposition $\varphi(F_1, \ldots, F_n) = fg + h$ is also the Euclidean division of $\varphi(F_1, \ldots, F_n)$ by f in the polynomial ring $K[T]$. By definition, $\varphi(F_1, \ldots, F_n) - u$ is divisible by f in $K[T]$. Therefore, the polynomial h is actually constant and equals u, which shows that $u \in k[S]$. □

Lemma 2.5 *Let A be a k-algebra which is an integral domain, and K the field of fractions of A. Let K'/K be a purely transcendental extension of transcendence degree* 1 *and B' a sub-k-algebra of finite type of K' which contains A. Then there exists a sub-k-algebra of finite type B of K which contains A.*

Proof Let $\alpha \in K'$ be a transcendental element over K such that $K' = K(\alpha)$. Assume that $B' = k[\varphi_1(\alpha), \ldots, \varphi_n(\alpha)]$, where each φ_i is a rational function of the form F_i/G_i, where F_i and G_i are polynomials of one variable with coefficients in K and $G_i \neq 0$. Let β be an element in the algebraic closure of the field K such that $G_i(\beta) \neq 0$ in $K'(\beta)$ for any $i \in \{1, \ldots, n\}$. Then one has $A \subset \widetilde{B} := k[\varphi_1(\beta), \ldots, \varphi_n(\beta)] \subset K(\beta)$.

In fact, if an element u of A can be written as $P(\varphi_1(\alpha),\ldots,\varphi_n(\alpha))$, where P is a polynomial with coefficients in k, then, since α is transcendental over $K(\beta)$, by considering α as the variable of rational functions and by specifying its value by β, we obtain that $u = P(\varphi_1(\beta),\ldots,\varphi_n(\beta))$. Finally, by applying Lemma 2.4 to $A \subset \widetilde{B}$ and the finite extension $K(\beta)/K$, we obtain that there exists a k-algebra of finite type $B \subset K$ such that $A \subset B$. □

Theorem 2.6 *Let k be a field and A be a k-algebra of subfinite type. We assume in addition that A is an integral domain and we denote by K the field of fractions of A. Then there exists a sub-k-algebra of finite type B of K such that $A \subset B$.*

Proof By Proposition 2.3, there exists a k-algebra of finite type B' which is an integral domain containing A. Let K' be the field of fractions of B', it is a finitely generated extension of K. Therefore there exists a sequence of extensions

$$K = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_n = K'$$

such that each extension K_i/K_{i-1} is generated by one element, $i \in \{1,\ldots,n\}$. The extension K_i/K_{i-1} is either generated by an algebraic element over K_{i-1} or is purely transcendental of transcendence degree 1. By induction we obtain that, for any $i \in \{0,\ldots,n-1\}$, there exists a sub-k-algebra of finite type B_i of K_i such that $B_i \supset A$. □

Corollary 2.7 *Let k be a field, R be an integral k-algebra of finite type and K be the fraction field of R. Let K' be an extension of k which is contained in K. Then there exists a finitely generated sub-k-algebra R' of K' containing $R \cap K'$, such that* $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$.

Proof By definition, $R \cap K'$ is an integral k-algebra of subfinite type. By Theorem 2.6, there exists a sub-k-algebra of finite type R' of $\mathrm{Frac}(R \cap K')$ such that $R \cap K' \subset R'$. Clearly one has $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$ since $R \cap K' \subset R' \subset \mathrm{Frac}(R \cap K')$. □

3 Graded linear series and subfiniteness

Let k be a field and K/k be a finitely generated field extension. Let

$$K[T] = \bigoplus_{n \in \mathbb{N}} KT^n$$

be the graded ring of polynomials of one variable with coefficients in K.

Definition 3.1 As a *graded linear series* of K/k we refer to a graded sub-k-algebra

$$V_\bullet = \bigoplus_{n \in \mathbb{N}} V_n T^n$$

of $K[T]$ such that $V_0 = k$ and that V_n is a finite dimensional k-vector subspace of K for any $n \in \mathbb{N}_{\geqslant 1}$.

Let $V_\bullet$ and $V'_\bullet$ be two graded linear series of K/k. If $V_n \subset V'_n$ for any $n \in \mathbb{N}$, we say that $V_\bullet$ *is contained in* $V'_\bullet$, or $V_\bullet$ *contains* $V'_\bullet$, and denote it by $V_\bullet \subset V'_\bullet$.

Let $V_\bullet$ be a graded linear series of K/k. If $V_\bullet$ is finitely generated as a k-algebra, we say that $V_\bullet$ is *of finite type*. If $V_\bullet$ is contained in a graded linear series of finite type, we say that it is *of subfinite type*. Note that graded linear series of subfinite type are also considered in [13] as *algebra of almost integral type*.

Let $V_\bullet$ be a graded linear series of K/k. We denote by $k(V_\bullet)$ the subextension of K/k generated by elements of the form f/g, where f and g are nonzero elements of K such that there exists $n \in \mathbb{N}_{\geqslant 1}$ with $f, g \in V_n$. The field $k(V_\bullet)$ is called *the field of rational functions of* $V_\bullet$.

Lemma 3.2 *Given any graded linear series $V_\bullet$ of K/k, one has*

$$k(V_n) = k(V_\bullet)$$

for every sufficiently large $n \in \mathbb{N}$ such that $V_n \neq \{0\}$, where $k(V_n)$ denotes the subextension of K/k generated by the elements of the form f/g with $\{f, g\} \subset V_n$, $g \neq 0$.

Proof First, we note that if $\ell \in \mathbb{N}_{\geqslant 1}$ is an index such that V_ℓ contains a nonzero element h, then $k(V_m) \subset k(V_{m+\ell n})$ for any $m, n \in \mathbb{N}_{\geqslant 1}$. In fact, if $\{f, g\} \subset V_m$ and $g \neq 0$, then

$$\frac{f}{g} = \frac{fh^n}{gh^n} \quad \text{and} \quad \{fh^n, gh^n\} \subset V_{m+\ell n}$$

for any $n \in \mathbb{N}_{\geqslant 1}$.

By changing the grading of $V_\bullet$, we may assume without loss of generality that $\{n \in \mathbb{N} : V_n \neq \{0\}\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module. There exist integers $\{n_1, \ldots, n_r\} \subset \mathbb{N}_{\geqslant 1}$ and nonzero elements $\{f_1, \ldots, f_r, g_1, \ldots, g_r\} \subset K$ such that $\{f_i, g_i\} \subset V_{n_i}$ for any $i \in \{1, \ldots, r\}$ and that

$$k(V_\bullet) = k(f_1/g_1, \ldots, f_r/g_r).$$

Set $p := \operatorname{lcm}(n_1, \ldots, n_r)$. By the above observation, we can assume $\{f_i, g_i\} \subset V_p$ for any i, and one has

$$k(V_\bullet) = k(f_1/g_1, \ldots, f_r/g_r) = k(V_p).$$

Moreover, by the hypothesis that $\{n \in \mathbb{N} : V_n \neq \{0\}\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module, we can find a positive integer q such that p and q are coprime and that $k(V_p) = k(V_q) = k(V_\bullet)$.

To conclude the proof, it suffices to show that $\{pm + qn : m, n \in \mathbb{N}\}$ contains every sufficiently large positive integer. Since p and q are coprime, we can fix $x, y \in \mathbb{Z}$ such that $px - qy = 1$. Moreover, we can assume that both x and y are positive. For any r with $0 \leqslant r < q$ and any n with $n \geqslant (q-1)y$,

$$qn + r = prx + q(n - ry) \in \{pm + qn : m, n \in \mathbb{N}\}.$$

Hence $\{pm + qn : m, n \in \mathbb{N}\}$ contains every integer not less than $q(q-1)y$. □

Remark 3.3 Let $V_\bullet$ be a graded linear series of K/k and f be a nonzero element of K. We denote by $V_\bullet(f)$ the graded linear series $\bigoplus_{n\in\mathbb{N}} f^n V_n T^n$, where $f^n V_n := \{f^n g : g \in V_n\}$, called the *twist of* $V_\bullet$ *by* f. Note that the twist does not change the field of rational functions: one has $k(V_\bullet(f)) = k(V_\bullet)$ for any $f \in K \setminus \{0\}$.

Proposition 3.4 *Let $W_\bullet$ be a graded linear series of finite type of K/k. Let n_0 be a positive integer. There exist an integer $r \geqslant 1$ and a family $(f_i T^{n_i})_{i=1}^r$ of homogeneous elements in $W_\bullet$ such that the following conditions are fulfilled:*

(i) *for any $i \in \{1, \dots, r\}$, one has $n_i \geqslant n_0$;*
(ii) *for any integer $n \geqslant n_0$, the vector space W_n is generated by elements of the form $f_1^{a_1} \cdots f_r^{a_r}$, where $a_1, \dots, a_r$ are natural numbers such that $a_1 n_1 + \cdots + a_r n_r = n$.*

Proof Suppose that $W_\bullet$ is generated by $W_1 T \oplus \cdots \oplus W_d T^d$. We claim that the graded linear series

$$k \oplus \bigoplus_{n \geqslant n_0} W_n T^n$$

is generated by $W_{n_0} T^{n_0} \oplus \cdots \oplus W_{2n_0+d-2} T^{2n_0+d-2}$. Let n be an integer such that $n \geqslant 2n_0 + d - 2$. Since $W_\bullet$ is generated by $W_1 T \oplus \cdots \oplus W_d T^d$, we obtain that

$$W_n = \sum_{\substack{(a_1,\dots,a_d)\in\mathbb{N}^d \\ a_1+2a_2+\cdots+da_d=n}} W_1^{a_1} \cdots W_d^{a_d}.$$

Let $(a_1, \dots, a_d)$ be an element in $\mathbb{N}^d$ such that $a_1 + 2a_2 + \cdots + da_d = n$. Since $n \geqslant 2n_0 + d - 2$, there exist an integer $m \geqslant 1$ and a family

$$\left\{ \left(a_1^{(i)}, \dots, a_d^{(i)}\right) : i \in \{1, \dots, m\} \right\}$$

of elements in $\mathbb{N}^d$ such that

$$a_j^{(1)} + \cdots + a_j^{(m)} = a_j \quad \text{for all} \quad j \in \{1, \dots, d\},$$
$$n_0 \leqslant a_1^{(i)} + 2a_2^{(i)} + \cdots + da_d^{(i)} \leqslant n_0 + d - 1 \quad \text{for all} \quad i \in \{1, \dots, m-1\},$$

and

$$n_0 \leqslant a_1^{(m)} + 2a_2^{(m)} + \cdots + da_d^{(m)} \leqslant 2n_0 + d - 2.$$

Therefore

$$W_n = \sum_{\substack{(b_{n_0},\dots,b_{2n_0+d-2})\in\mathbb{N}^{n_0+d-1} \\ n_0 b_{n_0}+\cdots+(2n_0+d-2)b_{2n_0+d-2}=n}} W_{n_0}^{b_{n_0}} \cdots W_{2n_0+d-2}^{b_{2n_0+d-2}},$$

which concludes the claim (b_j corresponds to the number of $i \in \{1, \dots, m\}$ such that $a_1^{(i)} + 2a_2^{(i)} + \cdots + da_d^{(i)} = j$). Finally it suffices to choose a family of homogeneous elements in $W_\bullet$ which forms a basis of $W_{n_0}T^{n_0} \oplus \cdots \oplus W_{2n_0+d-2}T^{2n_0+d-2}$. □

Lemma 3.5 *Let $K/k'/k$ be extensions of fields. We assume that the extension K/k is finitely generated and the extension k'/k is finite. Let $W'_\bullet$ be a graded linear series of finite type of K/k' and let*

$$W_\bullet = k \oplus \bigoplus_{n \in \mathbb{N}_{\geqslant 1}} W'_n T^n.$$

Then $W_\bullet$ is a graded linear series of finite type of K/k.

Proof Let $(f_i T^{n_i})_{i=1}^r$ be a system of generators of $W'_\bullet$. Let $(\theta_j)_{j=1}^m$ be a basis of k' over k. We claim that $W_\bullet$ is generated by

$$(\theta_j f_i T^{n_i})_{(i,j)\in\{1,\dots,r\}\times\{1,\dots,m\}}. \tag{3.1}$$

In fact, if φ is an element of W'_n, then it can be written as

$$\sum_{\substack{\boldsymbol{a}=(a_1,\dots,a_r)\in\mathbb{N}^r\\ a_1n_1+\cdots+a_rn_r=n}} \lambda_{\boldsymbol{a}} f_1^{a_1}\cdots f_r^{a_r},$$

where the coefficients $\lambda_{\boldsymbol{a}}$ belong to k'. By writing $\lambda_{\boldsymbol{a}}$ as a linear combination of $(\theta_j)_{j=1}^m$, we obtain that φ lies in the graded linear series of K/k generated by (3.1). □

Definition 3.6 Let $V_\bullet$ be a graded linear series of K/k. We assume that there exists $n \in \mathbb{N}_{\geqslant 1}$ such that $V_n \neq \{0\}$. We define the *Kodaira–Iitaka dimension* of $V_\bullet$ as the transcendence degree of $k(V_\bullet)$ over k. We refer the readers to [13, Section 3] and [6, Section 2] for the definition of Kodaira–Iitaka dimension in the setting of graded linear series of Cartier divisors or line bundles. If $V_n = \{0\}$ for any $n \in \mathbb{N}_{\geqslant 1}$, then by convention the *Kodaira–Iitaka dimension* of $V_\bullet$ is defined to be $-\infty$.

Theorem 3.7 *Let $V_\bullet$ be a graded linear series of K/k. Assume that there exists a graded linear series of finite type $V'_\bullet$ of K/k which contains $V_\bullet$. Then there exists a graded linear series of finite type $W_\bullet$ of K/k such that $V_\bullet \subset W_\bullet$ and $k(V_\bullet) = k(W_\bullet)$.*

Proof *Step 1: reduction to the case where* $1 \in V_1$ *and* $k(V'_1) = k(V'_\bullet)$. Let

$$\Theta := \{n \in \mathbb{N}_{\geqslant 1} : V_n \neq \{0\}\}.$$

The assertion of the theorem is trivial when $\Theta = \varnothing$. In the following, we assume that Θ is not empty, and hence it is a subsemigroup of $\mathbb{N}_{\geqslant 1}$. Let $a \in \mathbb{N}_{\geqslant 1}$ be a generator of the subgroup of $\mathbb{Z}$ generated by Θ. As $\bigoplus_{n\in\mathbb{N}} V'_{an}T^{an}$ is a k-algebra of finite type (see for example [9, Lemme II.2.1.6 (iv)]), by changing the grading we can reduce the problem to the case where $a = 1$. In particular, there exists an $m \in \mathbb{N}_{\geqslant 1}$ such that the vector spaces V_m and V_{m+1} are both nonzero. We pick $x \in V_m \setminus \{0\}$ and $y \in V_{m+1} \setminus \{0\}$. By replacing $V_\bullet$ by the graded linear series generated by $V_\bullet$ and $(y/x)T$ and replacing $V'_\bullet$ by the graded linear series generated by $V'_\bullet$ and $(y/x)T$ (this procedure does not change the fields of rational functions), we reduce the problem to the case where $V_1 \neq \{0\}$. Finally, by replacing $V_\bullet$ by $V_\bullet(f^{-1})$ and $V'_\bullet$ by $V'_\bullet(f^{-1})$ (see Remark 3.3 for the notation), where f is a nonzero element of V_1 (again this procedure does not change the fields of fractions, see Remark 3.3), we reduce the problem to the case where $1 \in V_1$. Moreover, by replacing $V'_\bullet$ by the graded linear series generated by $V'_\bullet$ and $\alpha_1 T, \ldots, \alpha_m T$, where $\{\alpha_1, \ldots, \alpha_m\}$ is a system of generators of $k(V'_\bullet)$ over k, we may assume that $k(V'_1) = k(V'_\bullet)$.

Step 2: reduction to the simple extension case by induction. As explained in the previous step, we can assume $1 \in V_1$ and $k(V'_1) = k(V'_\bullet)$. Since $k(V'_\bullet)/k(V_\bullet)$ is a finitely generated extension of fields (where V_1 is assumed to contain 1), there exist successive extensions of fields

$$k(V_\bullet) = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_b = k(V'_\bullet)$$

such that each extension K_i/K_{i-1} is generated by one element of V'_1.

Assume that the theorem has been proved for the case where $k(V'_\bullet)/k(V_\bullet)$ is generated by one element in V'_1. Then by induction we can show that, for any $i \in \{0, \ldots, b\}$, there exists a graded linear series of finite type $W^{(i)}_\bullet$, which contains $V_\bullet$ and such that $k(W^{(i)}_\bullet) = K_i$. In fact, we can choose $W^{(r)}_\bullet = V'_\bullet$. Assume that we have chosen a graded linear series of finite type $W^{(i+1)}_\bullet$ such that $W^{(i+1)}_\bullet \supset V_\bullet$ and $k(W^{(i+1)}_\bullet) = K_{i+1}$, where $i \in \{0, \ldots, b-1\}$. Let $V^{(i)}_\bullet$ be the graded linear series generated by $V_\bullet$ and a finite system of generators of K_i/k in V'_1. The graded linear series $V^{(i)}_\bullet$ contains $V_\bullet$ and $K_i = k(V^{(i)}_1)$. Without loss of generality we may assume that $V^{(i)}_\bullet \subset W^{(i+1)}_\bullet$ and that the extension K_{i+1}/K_i is generated by one element α in $W^{(i+1)}_1$, otherwise we just replace $W^{(i+1)}_\bullet$ by the graded linear series generated by $W^{(i+1)}_\bullet$, $V^{(i)}_1$ and a generator of the extension K_{i+1}/K_i in V'_1. It is a graded linear series of finite type which contains $V_\bullet$ and has K_{i+1} as its field of rational functions. If the theorem has been proved for the simple extension case, then we obtain the existence of a graded linear series of finite type $W^{(i)}_\bullet$ such that $V_\bullet \subset W_\bullet$ and $k(W^{(i)}_\bullet) = K_i$.

Note that the graded linear series $W_\bullet = W^{(0)}_\bullet$ satisfies the conditions $V_\bullet \subset W_\bullet$ and $k(V_\bullet) = k(W_\bullet)$. Therefore, to prove the theorem it suffices to prove the particular case where the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element in V'_1. Similarly, to prove the theorem under the supplementary condition that the extension $k(V'_\bullet)/k(V_\bullet)$ is algebraic, it suffices to prove the particular case where the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element in V'_1 which is algebraic over $k(V_\bullet)$.

Step 3: algebraic extension case. In this step, we prove the theorem under the assumption that the extension $k(V'_\bullet)/k(V_\bullet)$ is algebraic. As explained in the previous two steps, we may suppose without loss of generality that $1 \in V_1$, $k(V'_1) = k(V'_\bullet)$ and the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element α in V'_1 which is algebraic over $k(V_\bullet)$.

Let

$$G(X) := X^{\delta} + \xi_1 X^{\delta-1} + \cdots + \xi_{\delta} \in k(V_\bullet)[X]$$

be the minimal polynomial of α over $k(V_\bullet)$. By Proposition 3.4, there exist an integer $r \in \mathbb{N}_{\geqslant 1}$ and homogeneous elements $(f_i T^{n_i})_{i=1}^{r}$ with $n_i \geqslant \delta$ for any $i \in \{1, \ldots, r\}$, which generate the graded linear series

$$k \oplus \bigoplus_{n \geqslant \delta} V'_n T^n.$$

Since $1 \in V_n \subset V'_n$ for any $n \in \mathbb{N}_{\geqslant 1}$, for any $i \in \{1, \ldots, r\}$, one has $f_i \in k(V'_\bullet)$. Moreover, since the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by α (which is of degree δ over $k(V_\bullet)$), there exist polynomials

$$F_i(X) := \eta_{i,1} X^{\delta-1} + \cdots + \eta_{i,\delta} \in k(V_\bullet)[X], \quad i \in \{1, \ldots, r\},$$

such that $f_i = F_i(\alpha)$ for any $i \in \{1, \ldots, r\}$. We introduce the following polynomials in $k(V_\bullet)[T, Y]$:

$$\begin{aligned}
\widetilde{G}(T, Y) &= Y^{\delta} + (\xi_1 T) Y^{\delta-1} + \cdots + \xi_{\delta} T^{\delta}, \\
\widetilde{F}_i(T, Y) &= (\eta_{i,1} T^{n_i-\delta+1}) Y^{\delta-1} + \cdots + \eta_{i,\delta} T^{n_i}.
\end{aligned}$$

Note that one has $\widetilde{G}(T, TX) = G(X)T^{\delta}$ and $\widetilde{F}(T, TX) = F_i(X)T^{n_i}$.

We let $W_\bullet$ be the graded linear series generated by $V_1 T \oplus \cdots \oplus V_{\delta-1} T^{\delta-1}$ and the elements

$$\xi_1 T, \ldots, \xi_{\delta} T^{\delta}, \qquad \eta_{i,1} T^{n_i-\delta+1}, \ldots, \eta_{i,\delta} T^{n_i}, \quad i \in \{1, \ldots, r\}.$$

It is a graded linear series of finite type of K/k such that $k(W_\bullet) \subset k(V_\bullet)$. It remains to prove that $W_\bullet$ contains $V_\bullet$. Clearly $V_n \subset W_n$ for $n \in \{1, \ldots, \delta - 1\}$. Let $n \in \mathbb{N}_{\geqslant \delta}$ and φ be an element in $V_n \subset V'_n$. By definition φ can be written in the form

$$\sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1+\cdots+a_r n_r=n}} \lambda_{\boldsymbol{a}} f_1^{a_1} \cdots f_r^{a_r} \;=\; \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1+\cdots+a_r n_r=n}} \lambda_{\boldsymbol{a}} F_1(\alpha)^{a_1} \cdots F_r(\alpha)^{a_r},$$

where $\lambda_{\boldsymbol{a}} \in k$. We consider the element

$$\widetilde{F}(T, Y) \;=\; \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1+\cdots+a_r n_r=n}} \lambda_{\boldsymbol{a}} \widetilde{F}_1(T, Y)^{a_1} \cdots \widetilde{F}_r(T, Y)^{a_r} \in k(V_\bullet)[T, Y].$$

Viewed as a polynomial on Y with coefficients in $k(V_\bullet)[T]$, the coefficients of $\widetilde{F}(T, Y)$ can be written as the values of certain polynomials on

$$\eta_{i,1} T^{n_i-\delta+1}, \ldots, \eta_{i,\delta} T^{n_i}, \quad i \in \{1, \ldots, r\}.$$

Note that one has

$$\widetilde{F}(T, TX) = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1n_1+\cdot s+a_rn_r=n}} \lambda_{\boldsymbol{a}} F_1(X)^{a_1}\cdots F_r(X)^{a_r}T^n.$$

Therefore $\widetilde{F}(T, T\alpha) - \varphi T^n = 0$ in $k(V'_\bullet)[T]$. Since G is the minimal polynomial of α, a Euclidean division argument shows that φT^n can be written as a polynomial of $\xi_1 T, \ldots, \xi_\delta T^\delta$, $\eta_{i,1}T^{n_i-\delta+1}, \ldots, \eta_{i,\delta}T^{n_i}$, $i \in \{1, \ldots, r\}$, with coefficients in k. The theorem is thus proved in the particular case where $k(V'_\bullet)/k(V_\bullet)$ is an algebraic extension.

Step 4: general case. In this step, we prove the theorem in the general case. As explained in Steps 1 and 2, we may assume that $1 \in V_1$, $k(V'_1) = k(V'_\bullet)$ and that the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element α in V'_1 which is transcendental over $k(V_\bullet)$ (the algebraic case has already been treated in Step 3).

Since $V'_\bullet$ is of finite type, there exist an integer $r \geqslant 1$ and homogeneous elements $(f_iT^{n_i})_{i=1}^r$ which generate $V'_\bullet$ as a k-algebra. As $k(V'_\bullet)/k(V_\bullet)$ is generated by α, there exist rational functions P_i/Q_i, $i \in \{1, \ldots, r\}$, where $\{P_i, Q_i\} \subset k(V_\bullet)[X]$, $Q_i \neq 0$, such that $f_i = P_i(\alpha)/Q_i(\alpha)$.

Let θ be an element in the algebraic closure $\overline{k}$ of k, such that $Q_i(\theta) \neq 0$ for any $i \in \{1, \ldots, r\}$. Let $\widehat{k} = k(\theta)$ and $\widehat{K} = K(\theta)$. Then $\widehat{K}/K$ is a finite extension of field, and $\widehat{K}/\widehat{k}$ is a purely transcendental extension generated by α. Let $\widehat{V}_\bullet$ and $\widehat{V}'_\bullet$ be the graded sub-$\widehat{k}$-algebra of $\widehat{K}[T]$ generated by $V_\bullet$ and $V'_\bullet$ respectively. Then $\widehat{V}'_\bullet$ is generated as a $\widehat{k}$-algebra by $(f_iT^{n_i})_{i=1}^r$. We let $\widehat{W}_\bullet$ be the graded linear series of $\widehat{K}/\widehat{k}$ generated by T and elements of the form $(P_i(\theta)/Q_i(\theta))T^{n_i}$, where $i \in \{1, \ldots, r\}$. This is a graded linear series of finite type. Note that $P_i(\theta)/Q_i(\theta) \in \widehat{k}(\widehat{V}_\bullet)$ for any $i \in \{1, \ldots, r\}$. Therefore $\widehat{k}(\widehat{W}_\bullet) \subset \widehat{k}(\widehat{V}_\bullet)$.

Let $n \in \mathbb{N}_{\geqslant 1}$ and φ be an element of $\widehat{V}_n \subset \widehat{V}'_n$. By definition φ can be written in the form

$$\varphi = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1n_1+\cdots+a_rn_r=n}} \lambda_{\boldsymbol{a}} f_1^{a_1}\cdots f_r^{a_r},$$

where the coefficients $\lambda_{\boldsymbol{a}}$ belong to $\widehat{k}$. As α is transcendental over $\widehat{k}(\widehat{V}_\bullet)$, we obtain that

$$\varphi = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1n_1+\cdots+a_rn_r=n}} \lambda_{\boldsymbol{a}} \prod_{i=1}^r \left(\frac{P_i(\theta)}{Q_i(\theta)}\right)^{a_i},$$

which shows that $\varphi \in W_n$. Therefore one has $\widehat{V}_\bullet \subset \widehat{W}_\bullet$, which implies that $\widehat{k}(\widehat{V}_\bullet) = \widehat{k}(\widehat{W}_\bullet)$ since we have already seen that $\widehat{k}(\widehat{W}_\bullet) \subset \widehat{k}(\widehat{V}_\bullet)$.

Let

$$W'_\bullet := k \oplus \bigoplus_{n\in\mathbb{N}_{\geqslant 1}} \widehat{W}_n T^n.$$

Since $\widehat{W}_\bullet$ is a graded linear series of finite type of $\widehat{K}/\widehat{k}$, by Lemma 3.5 we obtain that $W'_\bullet$ is a graded linear series of $\widehat{K}/k$ of finite type. Moreover, one has $V_\bullet \subset W'_\bullet$ and $k(W'_\bullet) \subset \widehat{k}(\widehat{W}_\bullet) = \widehat{k}(\widehat{V}_\bullet)$ is a finite extension of $k(V_\bullet)$. Therefore, by the algebraic extension case of the theorem proved in Step 3 we obtain the existence of a graded linear series of finite type $W_\bullet$ of $\widehat{K}/k$ such that $V_\bullet \subset W_\bullet$ and that $k(V_\bullet) = k(W_\bullet)$. Moreover, the equality $k(V_\bullet) = k(W_\bullet)$ and the assumption $1 \in V_1 \subset W_1$ imply that $W_\bullet$ is a graded linear series of $k(V_\bullet)/k$ (and hence a graded linear series of K/k). $\square$

4 A subfinite version of Zariski's theorem

4.1 Preliminaries

In this section, we collect several basic facts on the valuations and on the graded rings, which we use to show Theorem 1.4.

4.1.1 Valuations

See Items 5 and 6 in the section of notation and conventions for definitions and basic notation related to valuations and their centres.

Lemma 4.1 *Let $\pi : X \to X'$ be a dominant morphism of integral separated k-schemes, $K := \mathrm{Rat}(X)$, $K' := \mathrm{Rat}(X')$, and ν be a discrete valuation of K over k. If the centre $c_X(\nu)$ of ν in X exists, then $\pi(c_X(\nu))$ is the centre of $\nu|_{K'}$ in X', namely $\pi(c_X(\nu)) = c_{X'}(\nu|_{K'})$.*

Proof Since the morphism π is dominant, it induces an injective homomorphism of fields $\mathrm{Rat}(X') \to \mathrm{Rat}(X)$, which allows to consider K' as a subfield of K. Recall that the centre $c_X(\nu)$ is the unique point $x \in X$ satisfying $\mathcal{O}_{X,x} \subset O_\nu$ and $\mathfrak{m}_x = \mathfrak{m}_\nu \cap \mathcal{O}_{X,x}$ (see notation and conventions 6). Note that

$$O_{\nu|_{K'}} = \{f \in K' : \nu(f) \geqslant 0\} = O_\nu \cap K', \quad \text{and} \quad \mathfrak{m}_{\nu|_{K'}} = \mathfrak{m}_\nu \cap K'.$$

Hence $O_{X',\pi(c_X(\nu))} \subset O_{\nu|_{K'}}$ and $\mathfrak{m}_{\pi(c_X(\nu))} \subset \mathfrak{m}_{\nu|_{K'}}$, which implies

$$\mathfrak{m}_{\pi(c_X(\nu))} = \mathfrak{m}_{\nu|_{K'}} \cap \mathcal{O}_{X',\pi(c_X(\nu))}$$

since $\mathfrak{m}_{\pi(c_X(\nu))}$ is a maximal ideal. $\square$

Lemma 4.2 *Let K/K' be a field extension of finite type. Then any discrete valuation ν' of K' extends to at least one discrete valuation ν of K such that the following diagram is commutative:*

$$\begin{tikzcd} K'^{\times} \arrow[r, "\nu'"] \arrow[d] & \mathbb{Q} \\ K^{\times} \arrow[ur, "\nu"'] & \end{tikzcd}$$

(*see notation and conventions* 5).

Proof By induction it suffices to treat the case where the extension K/K' is generated by one element α. If α is transcendental over K', then $K = K'(\alpha)$ is canonically isomorphic to the field of rational functions in one variable. Therefore the valuation $\nu\colon K \to \mathbb{Q} \cup \{+\infty\}$ such that

$$\nu(a_0 + a_1\alpha + \cdots + a_n\alpha^n) = \min\{\nu'(a_0), \ldots, \nu'(a_n)\}$$

for any $a_0 + a_1X + \cdots + a_nX^n \in K'[X]$ is a valuation extending ν'. The valuations ν' and ν have the same image and hence ν is discrete.

Assume that α is algebraic over K'. Let $\widehat{K'}$ be the completion of K' with respect to ν', on which the valuation ν' extends in a unique way. We choose an embedding of K in the algebraic closure $\widehat{K'}^{\mathrm{a}}$ of $\widehat{K'}$ and let L be the subfield of $\widehat{K'}^{\mathrm{a}}$ generated by $\widehat{K'}$ and K. Then L is a finite extension of $\widehat{K'}$, on which there is a unique valuation ω extending ν' such that

$$\omega(x) := \frac{1}{[K : \widehat{K'}]}\, \nu'\big(\mathrm{Norm}_{L/\widehat{K'}}(x)\big) \quad \text{for all } \ x \in L.$$

Let ν be the restriction of ω on K. It is a valuation extending ν'. Moreover, it is discrete since $\nu(K^\times) \subset \frac{1}{[K:\widehat{K'}]}\, \nu'(K'^\times)$. □

Lemma 4.3 *Let K/k be a field extension and let ν be any discrete valuation of K over k. Let $W_\bullet$ be a graded linear series of K/k of finite type and let $(f_iT^{d_i})_{i=1}^r$ be a system of generators of $W_\bullet$ over k. Set*

$$a := \min\left\{\frac{\nu(f_1)}{d_1}, \ldots, \frac{\nu(f_r)}{d_r}\right\}.$$

Then $W_n \subset \{\phi \in K : \nu(\phi) \geqslant na\}$ for every $n \in \mathbb{N}$.

Proof Any element in W_n can be written in the form

$$\sum_{d_1n_1+\cdots+d_rn_r=n} \alpha_{(n_1,\ldots,n_r)}\, f_1^{n_1}\cdots f_r^{n_r}, \quad \alpha_{(n_1,\ldots,n_r)} \in k.$$

Then

$$\nu\Bigg(\sum_{d_1n_1+\cdots+d_rn_r=n} \alpha_{(n_1,\ldots,n_r)}\, f_1^{n_1}\cdots f_r^{n_r}\Bigg) \geqslant \min\left\{\sum_{i=1}^r n_i\nu(f_i)\right\} \geqslant an.$$ □

4.1.2 Graded rings

Let $R_\bullet$ be a graded ring which is generated as R_0-algebra by a finite family of elements in R_1 and let $P := \mathrm{Proj}(R_\bullet)$. For each homogeneous element $a \in R_{\geqslant 1}$, let

$$(R_\bullet)_{(a)} := \left\{\frac{f}{a^p} : p \in \mathbb{N},\ \deg f = p \deg a\right\}$$

be the degree 0 component of the localisation $R_\bullet[1/a]$, and let

$$D_{\mathrm{Proj}(R_\bullet)+}(a) := \mathrm{Spec}\,(R_\bullet)_{(a)}$$

denote the affine open subscheme of $\mathrm{Proj}(R_\bullet)$ defined by the non-vanishing of a.

Set $\mathcal{O}_P(n) := \widetilde{R(n)_\bullet}$ (see notation and conventions 7). Given an $s \in R_n$, the local sections

$$s/1 \in H^0(D_{P+}(a), \mathcal{O}_P(n)) = (R(n)_\bullet)_{(a)}$$

for $a \in R_1$ glue up to a global section $\alpha_n(s) \in H^0(P, \mathcal{O}_P(n))$. The following lemmas are well known.

Lemma 4.4 ([9, Proposition II.2.7.3]) *Let $M_\bullet$ be a finitely generated graded $R_\bullet$-module. If $\widetilde{M_\bullet} = 0$, then $M_n = \{0\}$ for any sufficiently positive integer n.*

Lemma 4.5 *Let $R_\bullet$ be a graded ring and $P = \mathrm{Proj}(R_\bullet)$. If $R_\bullet$ is essentially integral and is generated as an R_0-algebra by finitely many homogeneous elements in R_1, then the canonical homomorphism*

$$\alpha_\bullet\colon R_\bullet \to R(\mathcal{O}_P(1))_\bullet := \bigoplus_{n\in\mathbb{N}} H^0(P, \mathcal{O}_P(n))$$

is injective and any element of $R(\mathcal{O}_P(1))_\bullet$ is integral over $R_\bullet$.

Proof Suppose that $R_\bullet$ is generated as an R_0-algebra by

$$\{a_1, \dots, a_r\} \subset R_1 \setminus \{0\},$$

where $a_1, \dots, a_r$ are all non-zerodivisors in $R_{\geqslant 1}$ since $R_\bullet$ is essentially integral (see notation and conventions 8). Given any $\mathfrak{p} \in P$, one can find an a_i such that $a_i \notin \mathfrak{p}$; hence $(D_{P+}(a_i))_{i\in\{1,\dots,r\}}$ covers P. Thus, a section in $R(\mathcal{O}_P(1))_\bullet$ can naturally be identified with an element in

$$\bigcap_{i=1}^{r} R_\bullet[1/a_i], \tag{4.1}$$

where the intersection is taken in $R_\bullet[1/(a_1 \dots a_r)]$. In particular, $\alpha_\bullet$ is injective.

Given any homogeneous element $u \in R(\mathcal{O}_P(1))_\bullet$, one can find an $e \geqslant 1$ such that $a_i^e u \in R_\bullet$ for every i by (4.1). Since $a_1, \dots, a_r$ generate $R_{\geqslant 1}$, one obtains $R_{\geqslant re} u \subset R_{\geqslant re}$. Moreover, by induction,

$$R_{\geqslant re} u^n \subset R_{\geqslant re} u^{n-1} \subset \cdots \subset R_{\geqslant re} u \subset R_{\geqslant re}$$

for every $n \geqslant 1$. It implies that $R_\bullet[u] \subset (1/a_1)^{re} R_\bullet$; hence u is integral over $R_\bullet$ (see for example [16, Theorem 9.1]). □

Lemma 4.6 *We keep the notation of Lemma 4.5. Suppose that $R_\bullet$ is a Noetherian integral domain and is generated as an R_0-algebra by finitely many homogeneous elements in R_1.*

(i) *If $R_\bullet$ is an N-1 ring, then there exists an $n_0 \geqslant 0$ such that α_n is isomorphic for every $n \geqslant n_0$.*
(ii) *If $R_\bullet$ is an integrally closed domain, then α_n is isomorphic for every $n \geqslant 0$.*

Proof (i) Recall that an integral domain is called an N-1 ring if its integral closure in its fraction field is a finite generated module over itself. Note that the graded rings $R_\bullet$ and $R'_\bullet := R(\mathcal{O}_P(1))_\bullet$ have the same homogeneous fraction field, which is the field of rational functions of the scheme $\mathrm{Proj}(R_\bullet)$. In particular, any homogeneous element of $R'_\bullet$ belongs to the homogeneous fraction field of $R_\bullet$, which is contained in the fraction field of $R_\bullet$. By Lemma 4.5 we obtain that $R'_\bullet$ is contained in the integral closure of $R_\bullet$ and hence is a module of finite type over $R_\bullet$ by the Noetherian and N-1 hypotheses.

We consider the exact sequence of $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}$-modules

$$0 \longrightarrow \widetilde{\mathrm{Ker}(\alpha_\bullet)} \longrightarrow \widetilde{R}_\bullet \xrightarrow{\widetilde{\alpha}_\bullet} \widetilde{R}'_\bullet \longrightarrow \widetilde{\mathrm{Coker}(\alpha_\bullet)} \longrightarrow 0.$$

Since $\widetilde{\alpha}_\bullet$ is isomorphic by [9, Proposition II.2.7.11], we have $\widetilde{\mathrm{Ker}(\alpha_\bullet)} = \widetilde{\mathrm{Coker}(\alpha_\bullet)} = 0$. Hence, by Lemma 4.4, we conclude.

(ii) If $R_\bullet$ is integrally closed, the above argument actually leads to $R_\bullet = R'_\bullet$ since $R'_\bullet$ is contained in the integral closure of $R_\bullet$. □

4.2 Proof of Theorem 1.4

Let X and X' be integral normal k-schemes with a fixed inclusion $\mathrm{Rat}(X') \subset \mathrm{Rat}(X)$. Each point $\xi \in X^{(1)} \cup X^{(0)}$ (respectively, $\xi' \in X'^{(1)} \cup X'^{(0)}$) defines the discrete valuation ord_ξ (respectively, $\mathrm{ord}_{\xi'}$) of $\mathrm{Rat}(X)$ (respectively, of $\mathrm{Rat}(X')$) over k. We define two sets of points on X and on X', respectively, as

$$\mathfrak{A}_{X/X'} := \left\{ \xi \in X^{(1)} \ : \ \begin{array}{l} \mathrm{ord}_\xi|_{\mathrm{Rat}(X')} \text{ is not equivalent to any} \\ \text{of } \mathrm{ord}_{\xi'} \text{ for } \xi' \in X'^{(1)} \cup X'^{(0)} \end{array} \right\}$$

and

$$\mathfrak{B}_{X/X'} := \left\{ \xi' \in X'^{(1)} \ : \ \begin{array}{l} \mathrm{ord}_{\xi'} \text{ is not equivalent to any} \\ \text{of } \mathrm{ord}_\xi|_{\mathrm{Rat}(X')} \text{ for } \xi \in X^{(1)} \end{array} \right\}.$$

Lemma 4.7 *Let X and X' be integral normal k-schemes of finite type with a fixed inclusion* $\mathrm{Rat}(X') \subset \mathrm{Rat}(X)$.

1. *The sets $\mathfrak{A}_{X/X'}$ and $\mathfrak{B}_{X/X'}$ are both finite.*
2. *If the inclusion* $\mathrm{Rat}(X') \subset \mathrm{Rat}(X)$ *is induced from a surjective and flat morphism $\pi : X \to X'$, then both $\mathfrak{A}_{X/X'}$ and $\mathfrak{B}_{X/X'}$ are empty.*
3. *If X' is proper over k and the inclusion* $\mathrm{Rat}(X') \subset \mathrm{Rat}(X)$ *is induced from a proper birational morphism $\pi : X \to X'$, then $\mathfrak{B}_{X/X'} = \varnothing$ and $\mathfrak{A}_{X/X'}$ is the set of the exceptional divisors of π.*

Proof 2: Let $\xi \in X^{(1)}$. Then by [10, Proposition IV.6.1.1] we have

$$\dim \mathcal{O}_{X',\pi(\xi)} = \dim \mathcal{O}_{X,\xi} - \dim \mathcal{O}_{\pi^{-1}(\pi(\xi)),\xi} = 0 \text{ or } 1.$$

Hence $\pi(\xi) \in X'^{(1)} \cup X'^{(0)}$ and $\mathrm{ord}_\xi|_{\mathrm{Rat}(X')}$ is equivalent to $\mathrm{ord}_{\pi(\xi)}$ by Lemma 4.1.

Let $\xi' \in X'^{(1)}$. Given any irreducible component Z of $\pi^{-1}(\overline{\{\xi'\}})$, the generic point ξ of Z is mapped to ξ' via π (see [10, Proposition IV.2.3.4]). Hence $\mathrm{ord}_{\xi'}$ is equivalent to $\mathrm{ord}_\xi|_{K'}$.

1: The inclusion $\mathrm{Rat}(X') \subset \mathrm{Rat}(X)$ yields a k-morphism $\pi : U \to X'$, where U denotes a nonempty open subscheme of X. By the theorem of generic flatness [10, Théorème IV.6.9.1], there exists a nonempty open subscheme $U' \subset X'$ such that

$$\overline{\pi} := \pi|_{\pi^{-1}(U')} : \overline{U} := \pi^{-1}(U') \to U'$$

is flat. Moreover, since $\overline{\pi}$ is an open morphism (see [10, Théorème IV.2.4.6]), we may assume that $\overline{\pi}$ is surjective. By the assertion 1 above, $\mathfrak{A}_{X/X'}$ (respectively, $\mathfrak{B}_{X/X'}$) is contained in the set consisting of the generic points of the irreducible components of $X \setminus \pi^{-1}(U')$ (respectively, $X' \setminus U'$).

3: By the valuative criterion of properness, there exists an open subscheme $U' \subset X'$ such that $\mathrm{codim}(X' \setminus U', X') \geqslant 2$ and the identification $\mathrm{Rat}(X') = \mathrm{Rat}(X)$ induces an open immersion $U' \to X$. Hence $\mathfrak{B}_{X/X'} = \varnothing$ and $\mathfrak{A}_{X/X'}$ is contained in the exceptional locus of π. If ξ is a generic point of an irreducible component of the exceptional locus of π, then $\pi(\xi) = c_{X'}(\mathrm{ord}_\xi|_{\mathrm{Rat}(X')})$ by Lemma 4.1 and $\dim \mathcal{O}_{X'!,\pi(\xi)}$ is $\geqslant 2$. Hence $\xi \in \mathfrak{A}_{X/X'}$. $\square$

We restate Theorem 1.4 as follows.

Theorem 4.8 *Let $K/K'/k$ be field extensions of finite type and $W_\bullet$ a graded linear series of K/k that is generated over k by the homogeneous elements of degree 1. We assume that W_1 contains $1 \in K$ and that the projective spectrum $P := \mathrm{Proj}(W_\bullet)$ is a normal scheme. Let X be any integral normal projective k-scheme whose field of rational functions is k-isomorphic to $k(W_\bullet \cap K'[T])$.*

1. *There then exists a $\mathbb{Q}$-Weil divisor D on X such that*

$$W_n \cap K' \subset H^0(X, nD) \subset k(W_\bullet \cap K'[T])$$

   for every sufficiently positive n.
2. *If $\mathfrak{A}_{P/X} = \varnothing$, then there exists a $\mathbb{Q}$-Weil divisor D on X such that*

$$W_n \cap K' = H^0(X, nD) \subset k(W_\bullet \cap K'[T])$$

   for every sufficiently positive n.

Proof Without loss of generality, we may assume that $K = k(W_\bullet)$ and $K' = \mathrm{Rat}(X)$. In particular, K naturally identifies with the field of rational functions on P. First, we give a valuation-theoretic interpretation of the required statement. Let H be the effective Cartier divisor on P defined by the image of 1 via $W_1 \to H^0(P, \mathcal{O}_P(1))$. By Lemma 4.6 (i), one has

$$\begin{aligned} W_n &= \{\phi \in K : nH + (\phi) \geqslant 0\} \\ &= \left\{\phi \in K : \mathrm{ord}_\xi(\phi) \geqslant -n\mathrm{mult}_\xi(H) \ \text{ for all } \ \xi \in P^{(1)}\right\} \end{aligned}$$

for every $n \gg 0$. Therefore,

$$W_n \cap K' = \left\{\phi \in K' : \mathrm{ord}_\xi|_{K'}(\phi) \geqslant -n\mathrm{mult}_\xi(H) \text{ for all } \xi \in P^{(1)}\right\} \tag{4.2}$$

for $n \gg 0$.

Next, for each $\xi' \in X^{(1)}$, we define a nonnegative rational number $a_{\xi'}$ as follows. If $\xi' \notin \mathfrak{B}_{P/X}$, then we fix an arbitrary point $\xi \in P^{(1)}$ such that $\mathrm{ord}_\xi|_{\mathrm{Rat}(X)}$ is equivalent to $\mathrm{ord}_{\xi'}$. Let e_ξ denote the ramification index of ord_ξ with respect to K/K' (see notation and conventions 5). We then set

$$a_{\xi'} := e_\xi^{-1}\mathrm{mult}_\xi(H).$$

Otherwise, we fix an arbitrary discrete valuation $\nu_{\xi'}$ of K extending $\mathrm{ord}_{\xi'}$, whose existence is assured by Lemma 4.2, and set

$$a_{\xi'} := -\min\{0, \nu_{\xi'}(f_1), \ldots, \nu_{\xi'}(f_r)\},$$

where $\{f_1T, \ldots, f_rT\}$ denotes a system of generators of $W_\bullet$ as a k-algebra. We define

$$D := \sum_{\xi' \in X^{(1)}} a_{\xi'}\overline{\{\xi'\}}.$$

By the finiteness of $\mathfrak{B}_{P/X}$ proved in Lemma 4.7, D is well defined as a $\mathbb{Q}$-Weil divisor on X. Moreover, D is effective and we have $W_n \cap K' \subset H^0(X, nD)$ for every $n \gg 0$ by (4.2) and Lemma 4.3.

Lastly, we consider the case where $\mathfrak{A}_{P/X} = \varnothing$. Given a $\xi' \in X^{(1)}$, we define a nonnegative rational number $b_{\xi'}$ as follows. If $\xi' \notin \mathfrak{B}_{P/X}$, then we set

$$b_{\xi'} := \min\left\{e_\xi^{-1}\mathrm{mult}_\xi(H) : \begin{array}{l} \xi \in P^{(1)}, e_\xi \neq 0, \text{ and } \mathrm{ord}_\xi|_{\mathrm{Rat}(X)} \text{ is} \\ \text{equivalent to } \mathrm{ord}_{\xi'} \end{array}\right\}.$$

Otherwise, we fix a discrete valuation $\nu_{\xi'}$ extending $\mathrm{ord}_{\xi'}$, and set

$$b_{\xi'} := -\min\{0, \nu_{\xi'}(f_1), \ldots, \nu_{\xi'}(f_r)\}$$

in the same way as above. If we set $D' := \sum_{\xi' \in X^{(1)}} b_{\xi'} \overline{\{\xi'\}}$, then, since $\mathfrak{A}_{P/X} = \varnothing$,

$$W_n \cap K' = \left\{\phi \in \mathrm{Rat}(X) : \mathrm{ord}_{\xi'}(\phi) \geqslant -nb_{\xi'} \text{ for all } \xi' \in X^{(1)} \setminus \mathfrak{B}_{P/X}\right\}$$
$$\supset H^0(X, nD')$$

for every $n \gg 0$. The reverse inclusion follows from the same argument as above. □

In the following, we give an alternative proof for Theorem 1.2 by using the projective version of Zariski's result (Theorem 1.4).

Corollary 4.9 *Let K/k be a finitely generated field extension and K'/k a subextension of K/k. Let $V_\bullet$ be a graded linear series of K'/k. If $V_\bullet$ is contained in a graded linear series $W_\bullet$ of K/k and of finite type over k, then $V_\bullet$ is contained a graded linear series $W'_\bullet$ of K'/k and of finite type over k.*

Proof We divide the proof into three steps.

Step 1: In this step, we make several reductions of the theorem. By the same arguments as in Step 1 of Theorem 3.7, we can assume that V_1 contains 1.

Claim 4.10 *By enlarging K if necessary, we can assume that $W_\bullet$ is generated by W_1 over k.*

Proof of Claim 4.10 Let $f_1 T^{d_1}, \dots, f_r T^{d_r} \in W_{\geqslant 1}$ be homogeneous generators of $W_\bullet$ over k. Let $T_1, \dots, T_r$ be variables with $\deg T_i = 1$ for every i. One can find a homogeneous prime ideal $\mathfrak{p}$ of $W_\bullet[T_1, \dots, T_r]$ such that $\mathfrak{p}$ contains

$$I := \left(T_1^{d_1} - f_1 T^{d_1}, \dots, T_r^{d_r} - f_r T^{d_r}\right)$$

and such that $\mathfrak{p} \cap V_\bullet = \{0\}$. In fact, let

$$W'_\bullet := W_\bullet[T_1, \dots, T_r]/I$$

and let a be a homogeneous element of degree $\geqslant 1$. Since the morphism

$$\mathrm{Spec}\,(W'_\bullet)_{(a)} \to \mathrm{Spec}\,(V_\bullet)_{(a)}$$

is dominant (Lemma 2.2), there exists a homogeneous prime ideal $\mathfrak{p} \in \mathrm{Proj}(W'_\bullet)$ such that $\mathfrak{p} \cap V_\bullet = \{0\}$. We set $U_\bullet := W'_\bullet/\mathfrak{p}$. Then $U_\bullet$ is a graded linear series of $k(U_\bullet)/k$, $W_\bullet \to U_\bullet$ is injective, and $U_\bullet$ is generated by

$$U_1 = W_1 + W_0 T_1 + \cdots + W_0 T_r.$$
■

In particular, we can assume that $P := \mathrm{Proj}(W_\bullet)$ is a projective scheme over k and that $\mathcal{O}_P(1) := \widetilde{W(1)_\bullet}$ is an invertible sheaf on P.

Step 2: Let $u: \widehat{P} \to P$ be a normalisation and H the Cartier divisor defined by the image of 1 via $V_1 \to H^0(\widehat{P}, u^*\mathcal{O}_P(1))$. We choose a very ample divisor $\widehat{H}$ such that $\widehat{H} - H$ is effective and such that $R(\widehat{H})_\bullet$ is generated by $R(\widehat{H})_1 T$ over $R(\widehat{H})_0$.

Note that the graded k-algebra

$$\widehat{W}_\bullet := k \oplus \bigoplus_{n\geqslant 1} H^0(\widehat{P}, n\widehat{H})T^n$$

is a graded linear series of K/k and of finite type over k (Lemma 3.5) and that $\mathrm{Proj}(\widehat{W}_\bullet)$ is isomorphic to $\widehat{P}$ over k.

Applying Theorem 1.4 to $\widehat{W}_\bullet$ and K'/k, we can find an integral normal projective k-scheme X, an effective $\mathbb{Q}$-divisor D on X, and an integer $n_0 \geqslant 1$ such that $\mathrm{Rat}(X) \subset K'$ and such that

$$V_n \subset R(\widehat{H})_n \cap K' \subset H^0(X, nD)$$

for every n with $n \geqslant n_0$.

Step 3: Let $\widehat{D}$ be a very ample divisor on X such that $\widehat{D} - D$ is effective and such that $R(\widehat{D})_\bullet$ is finitely generated over k. Let $W'_\bullet$ be the graded linear series generated by a basis of

$$\bigoplus_{n<n_0} V_n T^n$$

over k and by finite number of generators of $R(\widehat{D})_\bullet$ over k. Then W'_n contains V_n for every $n \geqslant 0$ and $W'_\bullet$ is finitely generated over k. □

As a consequence of Theorem 1.4, we can give an estimate of the following type for graded linear series of subfinite type (see also [14, Corollary 2.1.38] and Theorem 6.2 *infra*).

Corollary 4.11 *Let K/k be a finitely generated field extension and $V_\bullet$ be a graded linear series of K/k and of subfinite type. Let d be the Kodaira–Iitaka dimension of $V_\bullet$. If d is nonnegative, then there exist an integral normal projective k-scheme X and $\mathbb{Q}$-Cartier divisors D, D' on X such that the rational function field of X is k-isomorphic to $k(V_\bullet)$, that both D and D' have Kodaira–Iitaka dimension d, and that*

$$H^0(X, nD') \subset V_n \subset H^0(X, nD) \subset k(V_\bullet)$$

for every sufficiently positive n with $V_n \neq \{0\}$.

Proof The existence of D results from the same arguments as in Corollary 4.9. Thus, it suffices to show the existence of D' having the prescribed properties. By changing the grading of $V_\bullet$, we may assume that $\{n \in \mathbb{N} : V_n \neq \{0\}\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module. Choose any sufficiently positive integer p_0 such that $k(V_{p_0}) = k(V_\bullet)$ (see Lemma 3.2). Let $W_\bullet$ be the sub-k-algebra of $V_\bullet$ generated by V_{p_0}, and set

$$W'_\bullet := \bigoplus_{n\in\mathbb{N}} W_{p_0 n}.$$

Let $P := \mathrm{Proj}(W'_\bullet)$ and $\mathcal{O}_P(1) := \widetilde{W'_\bullet(1)}$. By Lemma 4.6, $W'_n = H^0(P, \mathcal{O}_P(n)) \subset V_{p_0 n}$ for every $n \gg 1$. Let $\nu\colon \widehat{P} \to P$ be a normalisation. Let p be any sufficiently positive integer divisible by p_0. Then one can find an ample divisor A on $\widehat{P}$ such that

$$H^0(\widehat{P}, nA) = H^0\big(P, \nu_*(\mathcal{O}_{\widehat{P}}(nA))\big) \subset H^0(P, \mathcal{O}_P(pn/p_0)) \subset V_{pn}$$

for every positive integer n (see the proof of [4, Proposition 3.6]).

Repeating the same arguments, one can choose an integral normal projective k-scheme X, two big Cartier divisors A, A' on X, and two coprime positive integers p, p' such that

$$H^0(X, nA) \subset V_{pn} \quad \text{and} \quad H^0(X, nA') \subset V_{p'n}$$

for any positive integer n. Moreover, one can choose an ample $\mathbb{Q}$-Cartier divisor D' on X and two coprime positive integers q, q' such that $qq'D'$ is integral, that q (resp. q') is divisible by p (resp. p'), and that

$$H^0(X, qnD') \subset H^0(X, (qn/p)A) \subset V_{qn}$$

and

$$H^0(X, q'nD') \subset H^0(X, (q'n/p')A) \subset V_{q'n}$$

hold for every integer $n \in \mathbb{N}_{\geqslant 1}$.

Since

$$H^0(X, qnD') \otimes_k H^0(X, q'n'D') \to H^0(X, (qn + q'n')D')$$

is surjective for any sufficiently positive integers n, n' (see for example [14, Example 1.2.22], which is valid over fields of arbitrary characteristics), we have $H^0(X, nD') \subset V_n$ for every sufficiently positive n (recall the arguments in Lemma 3.2). $\square$

Corollary 4.12 (Fujita [7, Appendix]) *Let X be an integral normal projective k-scheme and D an effective Cartier divisor on X. If the Kodaira–Iitaka dimension of D is 1, then the section ring $R(D)_\bullet$ is finitely generated.*

Proof Let $K := \mathrm{Rat}(X)$ and let C be the smooth projective k-curve with rational function field k-isomorphic to $K' := k(R(D)_\bullet)$. The inclusion $K' \subset K$ defines a rational map $X \dashrightarrow C$ and, by taking a suitable blow-up $\mu\colon \widehat{X} \to X$, one obtains a flat morphism $\pi : \widehat{X} \to C$ (the flatness follows from [11, Proposition III.9.7]). Note that $\mathfrak{A}_{\widehat{X}/C} = \mathfrak{B}_{\widehat{X}/C} = \varnothing$. If we set

$$E := \sum_{\xi' \in C^{(1)}} \min\big\{e_\xi^{-1}\mathrm{mult}_\xi(\mu^* D) : \xi \in \widehat{X}^{(1)},\ \xi \mapsto \xi',\ e_\xi \neq 0\big\}\,\xi',$$

then by Theorem 4.8,

$$H^0(C, nE) = H^0(\widehat{X}, n\mu^* D) = H^0(X, nD)$$

for every $n \gg 0$. Hence the result is reduced to the classic case of curves. □

Remark 4.13
- If X is a surface, Zariski [30] completely classified the cases where $R(D)_\bullet$ is finitely generated [30, Theorem 10.6 and Proposition 11.5]. Later, Fujita [7] generalised the case where the Kodaira–Iitaka dimension is one to the form of Theorem 4.12 by using the Iitaka fibrations.
- For a nef and big Cartier divisor D on X, $R(D)_\bullet$ is finitely generated if and only if D is semiample (see [14, Theorem 2.3.15]).

5 Nagata's counterexamples

In this section, we show how our results apply to Nagata's counterexamples. Let N and r be positive integers such that $N \geqslant r \geqslant 2$ and let $\boldsymbol{x} := (x_1, \ldots, x_N)$ and $\boldsymbol{y} := (y_1, \ldots, y_N)$ denote variables. Firstly, we consider the affine case as in [17–19,22]. Set

$$W_\bullet := \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}] = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_n,$$

where $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_n$ denotes the $\mathbb{C}$-vector space of the homogeneous polynomials of degree n in $(x_1, \ldots, x_N, y_1, \ldots, y_N)$, and let

$$K := \mathrm{Frac}(W_\bullet) = \mathbb{C}(\boldsymbol{x}, \boldsymbol{y})$$

be the fraction field of $W_\bullet$. Let

$$A = (a_{i,j})_{(i,j) \in \{1,\ldots,r\} \times \{1,\ldots,N\}}$$

be a matrix with coefficients in $\mathbb{C}$, where $r \in \{2, \ldots, N\}$. We assume that $a_{1,1} = \cdots = a_{1,N} = 1$ and that the block

$$(a_{i,j})_{(i,j) \in \{1,\ldots,r\} \times \{N-r+1,\ldots,N\}}$$

is invertible. Let $L_1, \ldots, L_r$ be a family of linear forms on $\mathbb{C}^N$ such that

$$\begin{pmatrix} L_1(t_1, \ldots, t_N) \\ \vdots \\ L_r(t_1, \ldots, t_N) \end{pmatrix} = A \begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix}.$$

Since A has r as its rank, the linear forms $L_1, \ldots, L_r$ are linearly independent. We introduce the following elements of $W_\bullet$ which are all homogeneous of degree N. We set $z_0 := x_1 \cdots x_N$ and for any $i \in \{1, \ldots, r\}$,

$$z_i := z_0 L_i\left(\frac{y_1}{x_1}, \ldots, \frac{y_N}{x_N}\right).$$

Let $K' := \mathbb{C}(z_0, z_1, \ldots, z_r)$. One has

$$K = K'(x_1, \ldots, x_{N-1}, y_1, \ldots, y_{N-r})$$

since

$$x_N = \frac{z_0}{x_1 \cdots x_{N-1}}$$

and, for any $i \in \{N - r + 1, \ldots, N\}$, y_i / x_i can be written as a linear form in

$$\frac{y_1}{x_1}, \quad \ldots, \quad \frac{y_{N-r}}{x_{N-r}}, \quad \frac{z_1}{z_0}, \quad \ldots, \quad \frac{z_r}{z_0}.$$

We denote by P_{r-1} the projective space $\operatorname{Proj}(\mathbb{C}[z_1, \ldots, z_r])$, and regard

$$\boldsymbol{a}_i := (a_{1,i} : \ldots : a_{r,i})$$

as a point in P_{r-1} for each $i \in \{1, \ldots, N\}$. Note that

$$H^0(P_{r-1}, \mathcal{O}_{P_{r-1}}(d)) = \mathbb{C}[z_1, \ldots, z_r]_d$$

for any integer $d \geqslant 0$. Each $W_n \cap K'$ is nonzero if and only if N divides n, and each element $F \in W_{Nn} \cap K'$ can be written in the form

$$z_0^{-m} f(z_1, \ldots, z_r),$$

where m is an integer and $f \in H^0(P_{r-1}, \mathcal{O}_{P_{r-1}}(m + n))$. In view of the following lemma, we know that the fraction field of $W_\bullet \cap K'$ coincides with K' and that $W_\bullet \cap K'$ is contained in $\mathbb{C}[z_0, z_1/z_0, \ldots, z_r/z_0]$.

Lemma 5.1 ([22, Lemma 3 (2)], [18, Lemma 2.45]) *Let $d \geqslant 1$ be any integer. For each $f \in H^0(P_{r-1}, \mathcal{O}_{P_{r-1}}(d))$ and for each $i \in \{1, 2, \ldots, N\}$, we have*

$$\operatorname{ord}_{\{x_i=0\}}|_{K'}(z_0^{-m} f(z_1, \ldots, z_r)) = \operatorname{ord}_{\boldsymbol{a}_i}(f(z_1, \ldots, z_r)) - m.$$

Nagata [22, p. 772] has conjectured the following: if $N \geqslant 10$ and $r = 3$, then, for generic N-points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N \in P_2$, one will have

$$\min(\operatorname{ord}_{\boldsymbol{a}_1}(f), \ldots, \operatorname{ord}_{\boldsymbol{a}_N}(f)) < \frac{d}{\sqrt{N}}$$

for every nonzero $f \in H^0(P_2, \mathcal{O}_{P_2}(d))$. We set

$$S_{(d,m)} := \left\{ z_0^{-m} f(z_1, z_2, z_3) \in W_\bullet \cap K' : f \in H^0(P_2, \mathcal{O}_{P_2}(d)) \right\}$$

for each $(d, m) \in \mathbb{N} \times \mathbb{Z}$, and set

$$\delta_m := \min\left(\frac{d}{m} : S_{(d,m)} \neq \{0\} \right)$$

for $m \geqslant 1$. If the conjecture is true, then one has $\delta_m > \sqrt{N}$ for every m and $\lim_{m \to \infty} \delta_m = \sqrt{N}$, which implies that the semigroup

$$\{(d, m) \in \mathbb{N} \times \mathbb{Z} : S_{(d,m)} \neq \{0\}\}$$

is not finitely generated. Hence

$$W_\bullet \cap K' = \bigoplus_{(d,m) \in \mathbb{N} \times \mathbb{Z}} S_{(d,m)}$$

is not of finite type over $\mathbb{C}$.

Nagata proved in [22, Section 3] that, if N is the square of an integer which is $\geqslant 4$, then the above conjecture is true. In particular, if $N = 16$, $r = 3$, and L_1, L_2, L_3 are generic, then $W_\bullet \cap K'$ is not of finite type over $\mathbb{C}$. Later, Mukai proved by applying Liouville's theorem that, if $N = 9$, $r = 3$, and L_1, L_2, L_3 are generic, then $W_\bullet \cap K'$ is not of finite type over $\mathbb{C}$ (see [18, Section 2.5]).

Next, we are going to consider a projective variant of Nagata's counterexample. Let T denote a variable for indicating the grading. We define a graded linear series of $K/\mathbb{C}$ as

$$\widehat{W}_\bullet := \mathbb{C}[T, \boldsymbol{x}T, \boldsymbol{y}T] = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_{\leqslant n} T^n,$$

where $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_{\leqslant n}$ denotes the $\mathbb{C}$-vector space of the polynomials of degree $\leqslant n$ in $(x_1, \ldots, x_N, y_1, \ldots, y_N)$. Note that, for each n, $F(\boldsymbol{x}, \boldsymbol{y}) \in \widehat{W}_n \cap K'$ if and only if $F(\boldsymbol{x}, \boldsymbol{y}) \in W_{\leqslant n} \cap K'$. Let $\widetilde{P} := \mathrm{Proj}(\widehat{W}_\bullet) \simeq \mathbb{P}^{2N}$ and

$$Q_{r+1} := \mathrm{Proj}\left(\mathbb{C}[T^N, z_0 T^N, z_1 T^N, \ldots, z_r T^N]\right) \simeq \mathbb{P}^{r+1}.$$

Let $H_r := \{T^N = 0\}$ and $D_r := \{z_0 T^N = 0\}$. Note that $H^0(Q_{r+1}, nH_r + nND_r)$ is the $\mathbb{C}$-vector space generated by

$$\left\{ z_0^{-m} f(z_1, \ldots, z_r) : f \in H^0(P_{r-1}, \mathcal{O}_{P_{r-1}}(m + n)),\ -n \leqslant m \leqslant nN \right\}.$$

Corollary 5.2 *We have $\widehat{W}_{Nn} \cap K' \subset H^0(Q_{r+1}, nH_r + nND_r)$ for every $n \in \mathbb{N}$.*

Remark 5.3 The following observations were suggested by one of the referees.

- It follows from Lemma 5.1 that $\mathfrak{A}_{\tilde{P}/Q_{r+1}}$ equals the set consisting of the generic points of $\{x_i = 0\}$ for $i \in \{1, \ldots, N\}$ and that $\mathfrak{B}_{\tilde{P}/Q_{r+1}}$ consists of the generic point of D_r. In particular, the Veronese subalgebra

$$\bigoplus_{n \in \mathbb{N}} (\widehat{W}_{Nn} \cap K')T^{Nn}$$

cannot be expressed as a complete linear series on Q_{r+1}. It remains an interesting question to see if the sets $\mathfrak{A}_{\tilde{P}/Q_{r+1}}$ and $\mathfrak{B}_{\tilde{P}/Q_{r+1}}$ contain also the information on the non-finite-generation property of the above graded linear series.
- If $r = 2$ and $N \geqslant 2$, then $\widehat{W}_\bullet \cap K'[T]$ is (finitely) generated by T, $z_0 T^N$, $z_1 T^N$, $z_2 T^N$, and

$$\frac{(a_{2,1}z_1 - z_2) \cdots (a_{2,N}z_1 - z_2)}{z_0} T^{N(N-1)}.$$

Hence, in this case, the volume of $\widehat{W}_\bullet \cap K'[T]$ is equal to $N^{-2}(N-1)^{-1}$ (see Definition 6.1 *infra*). Although the explicit computation of the volume function is in general a hard problem, it seems to us an intriguing question to obtain a combinatoric formula for the volume of $\widehat{W}_\bullet \cap K'[T]$ in the general case.
- By Fujita's approximation theorem in its graded linear series version (see [13, Corollary 3.11 (2)] and [15, Theorem 3.3 and Remark 3.4]), it is possible to approximate the graded linear series $\widehat{W}_\bullet \cap K'[T]$ by a family of amply polarised projective models of K'.

Remark 5.4 In [17,19], Mukai considered the subfield

$$K'' := \mathbb{C}(x_1, \ldots, x_N, z_1, \ldots, z_r)$$

and studied the finite generation of $\widehat{W}_\bullet \cap K''[T]$. In this case, we consider the weighted projective space

$$\mathrm{Proj}\big(\mathbb{C}[T, x_1T, \ldots, x_NT, z_1T^N, \ldots, z_rT^N]\big).$$

Let E_i (respectively, H) be the hyperplane defined by x_iT for $i \in \{1, 2, \ldots, N\}$ (respectively, T). One then has

$$\widehat{W}_\bullet \cap K''[T] \subset R(E_1 + \cdots + E_N + H)_\bullet,$$

where $R(E_1+\cdots+E_N+H)_\bullet$ denotes the total graded linear series of $E_1+\cdots+E_N+H$ (see notation and conventions 4).

6 Applications

In this section, we apply the subfinite criterion (Theorem 1.2) to the study of Fujita approximation for general subfinite graded linear series. Throughout the section, we let k be a field and K/k be a finitely generated field extension.

Definition 6.1 Let $V_\bullet$ be a graded linear series of K/k and d be its Kodaira–Iitaka dimension (see Definition 3.6). If $d \neq -\infty$, we define the *volume* of $V_\bullet$ as

$$\mathrm{vol}(V_\bullet) := \limsup_{n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!}.$$

A priori this invariant takes value in $[0, +\infty]$. We will see below that, if in addition the graded linear series $V_\bullet$ is of subfinite type (see Definition 3.1), then its volume is always a positive real number.

We say that a graded linear series $V_\bullet$ *satisfies the Fujita approximation property* if

$$\sup_{\substack{W_\bullet\subset V_\bullet\\ W_\bullet \text{ of finite type}\\ \dim(W_\bullet)=\dim(V_\bullet)}} \mathrm{vol}(W_\bullet) = \mathrm{vol}(V_\bullet),$$

where $W_\bullet$ runs over the set of all graded linear series of finite type which are contained in $V_\bullet$ and such that $W_\bullet$ has the same Kodaira–Iitaka dimension as $V_\bullet$.

The purpose of the section is to establish the following approximation result.

Theorem 6.2 *Any graded linear series $V_\bullet$ of K/k which is of subfinite type and has nonnegative Kodaira–Iitaka dimension d satisfies the Fujita approximation property. Moreover, one has*

$$\mathrm{vol}(V_\bullet) = \lim_{\substack{n\in\mathbb{N}(V_\bullet)\\ n\to+\infty}} \frac{\dim_k(V_n)}{n^d/d!} \in (0, +\infty),$$

where $\mathbb{N}(V_\bullet) = \{n \in \mathbb{N} : V_n \neq \{0\}\}$.

Proof By changing the grading we may assume without loss of generality that $V_n \neq \{0\}$ for sufficiently positive integer n. Let K' be the homogeneous fraction field $k(V_\bullet)$. Note that K'/k is a subextension of K/k and hence is finitely generated. Moreover, by Theorem 1.2, we obtain that $V_\bullet$ viewed as a graded linear series of K'/k is of subfinite type. Therefore, the assertions follow from [5, Theorem 1.1] (by definition $V_\bullet$ is birational if we consider it as a graded linear series of K'). □

Remark 6.3 In the case where the field K admits a valuation *of one-dimensional leaves* in a totally ordered abelian group of finite type (this is the case notably when k is an algebraically closed field), we recover a result of Kaveh and Khovanskii [13, Corollary 3.11 (2)]. Note that the existence of a valuation of one-dimensional leaves on $V_\bullet$ implies that $V_\bullet$ is geometrically integral since such a valuation induces by extension of scalars a valuation of one-dimensional leaves on $V_\bullet \otimes_k k'$ for any extension of fields

k'/k. In particular, for any pair of homogeneous elements x and y of $V_\bullet \otimes_k k'$, the valuation of xy is equal to the sum of the valuations of x and y, which implies that $V_\bullet \otimes_k k'$ is an integral domain.

By combining the results of [4] and the subfiniteness result (Theorem 1.2), we obtain the following upper bound for the Hilbert–Samuel function of general graded linear series of subfinite type.

Theorem 6.4 *Let $V_\bullet$ be a graded linear series of K/k and d its Kodaira–Iitaka dimension. There then exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that*

$$f(n) = \mathrm{vol}(V_\bullet)\,\frac{n^d}{d!} + \mathrm{O}(n^{d-1}), \quad n \to +\infty,$$

and

$$\dim_k(V_n) \leqslant f(n) \quad \textit{for all} \;\; n \in \mathbb{N}.$$

Remark 6.5 The result [5, Theorem 1.1] actually provides more geometric information about the graded linear series of subfinite type. Let K/k be a finitely generated transcendental field extension and let d be the transcendence degree of K/k. We fix a flag

$$k = K_0 \subset K_1 \subset \cdots \subset K_d = K$$

of subfields of K containing k such that each extension K_i/K_{i-1} is transcendental and has transcendence degree 1. Let $\mathcal{A}(K/k)$ be the set of all graded linear series of subfinite type $V_\bullet$ of K/k such that $k(V_\bullet) = k$. Then there has been constructed in [5] a map Δ from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$ which satisfies the following conditions:

(a) If $V_\bullet$ and $V'_\bullet$ are two graded linear series in $\mathcal{A}(K/k)$ such that $V_\bullet \subset V'_\bullet$, then one has $\Delta(V_\bullet) \subset \Delta(V'_\bullet)$.

(b) If $V_\bullet$ and $W_\bullet$ are two graded linear series in $\mathcal{A}(K/k)$, then

$$\Delta(V_\bullet \cdot W_\bullet) \supset \Delta(V_\bullet) + \Delta(W_\bullet) := \{x + y : x \in \Delta(V_\bullet),\ y \in \Delta(W_\bullet)\},$$

where $V_\bullet \cdot W_\bullet$ denotes the graded linear series whose n-th homogeneous component is the k-vector space generated by $\{fg : f \in V_n,\ g \in W_n\}$.

(c) For any graded linear series $V_\bullet$ in $\mathcal{A}(K/k)$, the volume of $V_\bullet$ identifies with the Lebesgue measure of $\Delta(V_\bullet)$ multiplied by $d!$.

This allows us to construct the arithmetic analogue of Newton–Okounkov bodies for general arithmetic graded linear series of subfinite type, using the ideas of [1].

In what follows, we assume that k is a number field. We denote by M_k the set of all places of k. For each $v \in M_k$, let $|\cdot|_v$ be an absolute value on k which extends either the usual absolute value or certain p-adic absolute value (so that $|p|_v = p^{-1}$) on $\mathbb{Q}$.

As *adelic vector bundle* on $\operatorname{Spec} k$, we refer to the data $\overline{V} = (V, (\|\cdot\|_v)_{v\in M_k})$ of a finite dimensional vector space V over k and a family of norms $\|\cdot\|_v$ over $V\otimes_k k_v$ such that there exists a basis $(e_i)_{i=1}^r$ of V over k and a finite subset S of M_k satisfying the following condition:

$$\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_v = \max_{i\in\{1,\dots,r\}} |\lambda_i|_v \quad \text{for all} \quad v \in M_k \setminus S \quad \text{and} \quad (\lambda_1, \dots, \lambda_r) \in k_v^r.$$

Given an adelic vector bundle $\overline{V}$ on $\operatorname{Spec} k$, for any nonzero element $s \in V$, we define the *Arakelov degree of* s as

$$\widehat{\deg}(s) := -\sum_{v\in M_k} [k_v : \mathbb{Q}_v] \ln \|s\|_v.$$

By the product formula

$$\sum_{v\in M_k} [k_v : \mathbb{Q}_v] \ln |a|_v = 0 \quad \text{for all} \quad a \in k^\times,$$

we obtain that

$$\widehat{\deg}(as) = \widehat{\deg}(s) \quad \text{for all} \quad a \in k^\times.$$

Moreover, the *Arakelov degree of* $\overline{V}$ is defined as

$$-\sum_{v\in M_k} \ln \|\eta\|_{v,\det},$$

where η is a nonzero element of $\det(V)$, and

$$\|\eta\|_{v,\det} = \inf\left\{ \|x_1\|_v \cdots \|x_r\|_v \, : \, \eta = x_1 \wedge \cdots \wedge x_r \right\}.$$

Again by the product formula we obtain that the definition does not depend on the choice of $\eta \in \det(V) \setminus \{0\}$.

Let $\overline{V}$ be an adelic vector bundle of rank r on $\operatorname{Spec} k$. For any $t \in \mathbb{R}$, let

$$\mathcal{F}^t(V) = \operatorname{Vect}_k(\{s \in V \setminus \{0\} : \widehat{\deg}(s) \geqslant t\}).$$

This is a decreasing $\mathbb{R}$-filtration on V, called the $\mathbb{R}$-*filtration by minima*. Note that for any $i \in \{1, \dots, r\}$, the number

$$\lambda_i(\overline{V}) = \sup\{t \in \mathbb{R} : \operatorname{rk}_k(\mathcal{F}^t(V)) \geqslant i\}$$

coincides with the minus logarithmic version of the i-th minima in the sense of Roy and Thunder [26,27]. For any $s \in V$, we let

$$\lambda(s) := \sup\{t \in \mathbb{R} : s \in \mathcal{F}^t(V)\}.$$

In the following, we let K/k be a finitely generated field extension of the number field k. Let $V_\bullet$ be a graded linear series of subfinite type of K/k. For each $n \in \mathbb{N}$, we equip V_n with a structure of adelic vector bundle $(V_n, (\|\cdot\|_{n,v})_{v\in M_k})$ on $\operatorname{Spec} k$ so that, for any $v \in M_k$,

$$\|s_n \cdot s_m\|_v \leqslant \|s_n\|_v \cdot \|s_m\|_v \quad \text{for all} \quad (n,m) \in \mathbb{N}^2 \quad \text{and} \quad (s_n, s_m) \in V_n \times V_m. \tag{6.1}$$

We assume in addition that

$$\lambda_{\max}(\overline{V}_\bullet) := \limsup_{n\to+\infty} \frac{\lambda_1(\overline{V}_n)}{n} < +\infty.$$

This condition implies that $V_\bullet$ has a nonnegative Kodaira–Iitaka dimension. For any $t \in \mathbb{R}$, let

$$V_\bullet^t := \bigoplus_{n\in\mathbb{N}} \mathcal{F}^{nt}(V_n).$$

It is a graded linear series of K/k. By definition one has $V_n^t = \{0\}$ if $n \in \mathbb{N}_{\geqslant 1}$ and $t > \lambda_{\max}(\overline{V}_\bullet)$.

Proposition 6.6 *For any $t < \lambda_{\max}(\overline{V}_\bullet)$, one has $k(V_\bullet) = k(V_\bullet^t)$.*

Proof Clearly one has $k(V_\bullet) \supset k(V_\bullet^t)$. It suffices to prove the converse inclusion. Let $n \geqslant 1$ be an integer and f, g be nonzero elements in V_n. Since $t < \lambda_{\max}(V_\bullet)$ there exist $m \in \mathbb{N}_{\geqslant 1}$ and $s \in V_m$ such that $\lambda(s) > mt$. Thus for sufficiently positive integer ℓ one has $\lambda(s^\ell f) > (\ell m + n)t$ and $\lambda(s^\ell g) > (\ell m + n)t$. Therefore $\{s^\ell f, s^\ell g\} \subset V_{\ell m+n}^t$, which implies $f/g \in k(V_\bullet^t)$. □

The above proposition allows us to consider $V_\bullet^t$ as a birational graded linear series of $k(V_\bullet)/k$ and to construct its Newton–Okounkov body as recalled in Remark 6.5. We define the *concave transform* of $\overline{V}_\bullet$ as the function $G_{\overline{V}_\bullet}$ on $\Delta(V_\bullet)$ sending $x \in \Delta(V_\bullet)$ to

$$\sup\{t < \lambda_{\max}(\overline{V}_\bullet) : x \in \Delta(V_\bullet^t)\}.$$

By the condition (b) in Remark 6.5, the function $G_{\overline{V}_\bullet}$ is concave.

The following result generalises [1, Theorem 2.8] to the case of subfinite adelically normed graded linear series.

Theorem 6.7 *Let K/k be a finitely generated extension of a number field k, and $\overline{V}_\bullet = \bigoplus_{n\in\mathbb{N}} \overline{V}_n$ a graded linear series of subfinite type of K/k of Kodaira–Iitaka dimension $d \geqslant 0$, equipped with structures of adelic vector bundles on* Spec *k, which satisfy the submultiplicativity condition (6.1) and the condition $\lambda_{\max}(\overline{V}_\bullet) < +\infty$. Then the sequence of measures*

$$\frac{1}{\operatorname{rk}_k(V_n)} \sum_{i=1}^{\operatorname{rk}_k(V_n)} \delta_{\lambda_i(\overline{V}_n)/n}, \quad n \in \mathbb{N}(V_\bullet) = \{m \in \mathbb{N} : V_m \neq \{0\}\},$$

converges weakly to a Boreal probability measure on $\mathbb{R}$*, which is the image of the uniform measure*

$$\frac{1}{\operatorname{vol}(\Delta(V_\bullet))}\,\mathbb{1}_{\Delta(V_\bullet)}(x)\,\mathrm{d}x$$

by the concave transform $G_{\overline{V}_\bullet}$.

Proof For any $t < \lambda_{\max}(\overline{V}_\bullet)$, the graded linear series $V_\bullet^t$ has the same homogeneous fraction field as $V_\bullet$ (see Proposition 6.6). Hence we can construct a decreasing family $(\Delta(V_\bullet^t))_{t<\lambda_{\max}(\overline{V}_\bullet)}$ of convex bodies contained in $\Delta(V_\bullet)$, as described in Remark 6.5. Moreover, if t_1 and t_2 are two real numbers which are $< \lambda_{\max}(\overline{V}_\bullet)$. Then, by the same method as in [1, Section 1.3], we obtain the desired result. $\square$

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