POSITIVE DEGREE AND ARITHMETIC BIGNESS

Huayi Chen

Abstract. — We establish, for a generically big Hermitian line bundle, the convergence of truncated Harder-Narasimhan polygons and the uniform continuity of the limit. As applications, we prove a conjecture of Moriwaki asserting that the arithmetic volume function is actually a limit instead of a sup-limit, and we show how to compute the asymptotic polygon of a Hermitian line bundle, by using the arithmetic volume function.

Contents

1. Introduction

Let K be a number field and $\mathcal{O}_K$ be its integer ring. Let $\mathcal{X}$ be a projective arithmetic variety of total dimension d over $\operatorname{Spec}\mathcal{O}_K$. For any Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$, the *arithmetic volume* of $\overline{\mathcal{L}}$ introduced by Moriwaki (see [**20**]) is

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \limsup_{n\to\infty} \frac{\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})}{n^d/d!}, \tag{1}$$

huayi.chen@polytechnique.org.

where $\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n}) := \log\#\{s\in H^0(\mathcal{X},\mathcal{L}^{\otimes n})\mid \forall\sigma : K\to\mathbb{C},\ \|s\|_{\sigma,\sup}\leq 1\}$. The Hermitian line bundle $\overline{\mathcal{L}}$ is said to be *arithmetically big* if $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})>0$. The notion of arithmetic bigness had been firstly introduced by Moriwaki [**19**] §2 in a different form. Recently he himself ([**20**] Theorem 4.5) and Yuan ([**25**] Corollary 2.4) have proved that the arithmetic bigness in [**19**] is actually equivalent to the strict positivity of the arithmetic volume function (1). In [**20**], Moriwaki has proved the continuity of (1) with respect to $\overline{\mathcal{L}}$ and then deduced some comparisons to arithmetic intersection numbers (*loc. cit.* Theorem 6.2).

Note that the volume function (1) is an arithmetic analogue of the classical volume function for line bundles on a projective variety: if L is a line bundle on a projective variety X of dimension d defined over a field k, the *volume* of L is

$$\mathrm{vol}(L) := \limsup_{n\to\infty}\frac{\mathrm{rk}_k\, H^0(X,L^{\otimes n})}{n^d/d!}. \tag{2}$$

Similarly, L is said to be *big* if $\mathrm{vol}(L)>0$. After Fujita's approximation theorem (see [**13**], and [**23**] for positive characteristic case), the sup-limit in (2) is in fact a limit (see [**18**] 11.4.7).

During a presentation at *Institut de Mathématiques de Jussieu*, Moriwaki has conjectured that, in arithmetic case, the sequence $\big(\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}})/n^d\big)_{n\geq 1}$ also converges. In other words, one has actually

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \lim_{n\to\infty}\frac{\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})}{n^d/d!}.$$

The strategy proposed by him is to develop an analogue of Fujita's approximation theorem in arithmetic setting (see [**20**] Remark 5.7).

In this article, we prove Moriwaki's conjecture by establishing a convergence result of Harder-Narasimhan polygons (Theorem 4.2), which is a generalization of the author's previous work [**11**] where the main tool was the Harder-Narasimhan filtration (indexed by $\mathbb{R}$) of a Hermitian vector bundle on $\mathrm{Spec}\,\mathcal{O}_K$ and its associated Borel measure. To apply the convergence of polygons, we shall compare $\widehat{h}^0(\overline{E})$, defined as the logarithm of the number of effective points in E, to the *positive degree* $\widehat{\deg}_+(\overline{E})$, which is the maximal value of the Harder-Narasimhan polygon of $\overline{E}$. Here $\overline{E}$ denotes a Hermitian vector bundle on $\mathrm{Spec}\,\mathcal{O}_K$. We show that the arithmetic volume function coincides with the limit of normalized positive degrees and therefore prove the conjecture.

In [**20**] and [**25**], the important (analytical) technic used by both authors is the estimation of the distortion function, which has already appeared in [**1**]. The approach in the present work, which is similar to that in [**21**], relies on purely algebraic arguments. We also establish an explicit link between the volume function and some geometric invariants of $\overline{\mathcal{L}}$ such as asymptotic slopes, which permits us to prove that $\overline{\mathcal{L}}$ is big if and only if the norm of the smallest non-zero section of $\overline{\mathcal{L}}^{\otimes n}$ decreases exponentially when n tends to infinity. This result is analogous to Theorem 4.5 of [**20**] or Corollary 2.4 (1)⇔(4) of [**25**] except that we avoid using analytical methods.

In our approach, the arithmetic volume function can be interpreted as the limit of maximal values of Harder-Narasimhan polygons. Inspired by Moriwaki's work [**20**], we shall establish the uniform continuity for limit of truncated Harder-Narasimhan polygons (Theorem 6.4). This result refines *loc. cit.* Theorem 5.4. Furthermore, we show that the asymptotic polygon can be calculated from the volume function of the Hermitian line bundle twisted by pull-backs of Hermitian line bundles on $\operatorname{Spec}\mathcal{O}_K$.

Our method works also in function field case. It establishes an explicit link between the geometric volume function and some classical geometry such as semistability and Harder-Narasimhan filtration. This generalizes for example a work of Wolfe [**24**] (see also [**12**] Example 2.12) concerning volume function on ruled varieties over curves. Moreover, recent results in [**7, 8, 2**] show that at least in function field case, the asymptotic polygon is "differentiable" with respect to the line bundle, and there may be a "measure-valued intersection product" from which we recover arithmetic invariants by integration.

The rest of this article is organized as follows. We fist recall some notation in Arakelov geometry in the second section. In the third section, we introduce the notion of positive degree for a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$ and we compare it to the logarithm of the number of effective elements. The main tool is the Riemann-Roch inequality on $\operatorname{Spec}\mathcal{O}_K$ due to Gillet and Soulé [**15**]. In the fourth section, we establish the convergence of the measures associated to suitably filtered section algebra of a big line bundle (Theorem 4.2). We show in the fifth section that the arithmetic bigness of $\overline{\mathcal{L}}$ implies the classical one of $\mathcal{L}_K$. This gives an alternative proof for a result of Yuan [**25**]. By the convergence result in the fourth section, we are able to prove that the volume of $\overline{\mathcal{L}}$ coincides with the limit of normalized positive degrees, and therefore the sup-limit in (1) is in fact a limit (Theorem 5.2). Here we also need the comparison result in the third section. Finally, we prove that the arithmetic bigness is equivalent to the positivity of asymptotic maximal slope (Theorem 5.4). In the sixth section, we establish the continuity of the limit of truncated polygons. Then we show in the seventh section how to compute the asymptotic polygon.

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2. Notation and reminders

Throughout this article, we fix a number field K and denote by $\mathcal{O}_K$ its algebraic integer ring, and by Δ_K its discriminant. By (projective) *arithmetic variety* we mean an integral projective flat $\mathcal{O}_K$-scheme.

2.1. Hermitian vector bundles. — If $\mathcal{X}$ is an arithmetic variety, one calls *Hermitian vector bundle* on $\mathcal{X}$ any pair $\overline{\mathcal{E}} = (\mathcal{E}, (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, and for any embedding $\sigma : K \to \mathbb{C}$, $\|\cdot\|_\sigma$ is a continuous Hermitian norm on $\mathcal{E}_{\sigma,\mathbb{C}}$. One requires in addition that the metrics $(\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$ are invariant by the action of complex conjugation. The *rank* of $\overline{\mathcal{E}}$ is just that of $\mathcal{E}$. If $\operatorname{rk}\mathcal{E} = 1$, one says that $\overline{\mathcal{E}}$ is a *Hermitian line bundle*. Note that $\operatorname{Spec}\mathcal{O}_K$ is itself an arithmetic variety. A Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$ is just a projective $\mathcal{O}_K$-module equipped with Hermitian norms which are invariant under complex conjugation. For any real number a, denote by $\overline{\mathscr{L}}_a$ the Hermitian line bundle

$$\overline{\mathscr{L}}_a := (\mathcal{O}_K, (\|\cdot\|_{\sigma,a})_{\sigma:K\to\mathbb{C}}), \tag{3}$$

where $\|\mathbf{1}\|_{\sigma,a} = e^{-a}$, $\mathbf{1}$ being the unit of $\mathcal{O}_K$.

2.2. Arakelov degree, slope and Harder-Narasimhan polygon. — Several invariants are naturally defined for Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, notably the *Arakelov degree*, which leads to other arithmetic invariants (cf. **[4]**). If $\overline{E}$ is a Hermitian vector bundle of rank r on $\operatorname{Spec}\mathcal{O}_K$, the *Arakelov degree* of $\overline{E}$ is defined as the real number

$$\widehat{\deg}(\overline{E}) := \log\#\big(E/(\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r)\big) - \frac{1}{2}\sum_{\sigma:K\to\mathbb{C}} \log\det\big(\langle s_i, s_j\rangle_\sigma\big)_{1\le i,j\le r},$$

where $(s_i)_{1\le i\le r}$ is an element in E^r which forms a basis of E_K. This definition does not depend on the choice of $(s_i)_{1\le i\le r}$. If E is non-zero, the *slope* of $\overline{E}$ is defined to be the quotient $\widehat{\mu}(\overline{E}) := \widehat{\deg}(\overline{E})/\operatorname{rk}E$. The *maximal slope* of $\overline{E}$ is the maximal value of slopes of all non-zero Hermitian subbundles of $\overline{E}$. The *minimal slope* of $\overline{E}$ is the minimal value of slopes of all non-zero Hermitian quotients of $\overline{E}$. We say that $\overline{E}$ is *semistable* if $\widehat{\mu}(\overline{E}) = \widehat{\mu}_{\max}(\overline{E})$.

Recall that the *Harder-Narasimhan polygon* $P_{\overline{E}}$ is by definition the concave function defined on $[0, \operatorname{rk}E]$ whose graph is the convex hull of points of the form $(\operatorname{rk}F, \widehat{\deg}(\overline{F}))$, where $\overline{F}$ runs over all Hermitian subbundles of $\overline{E}$. By works of Stuhler **[22]** and Grayson **[16]**, this polygon can be determined from the Harder-Narasimhan flag of $\overline{E}$, which is the only flag

$$E = E_0 \supset E_1 \supset \cdots \supset E_n = 0 \tag{4}$$

such that the subquotients $\overline{E}_i/\overline{E}_{i+1}$ are all semistable, and verifies

$$\widehat{\mu}(\overline{E}_0/\overline{E}_1) < \widehat{\mu}(\overline{E}_1/\overline{E}_2) < \cdots < \widehat{\mu}(\overline{E}_{n-1}/\overline{E}_n). \tag{5}$$

In fact, the vertices of $P_{\overline{E}}$ are just $(\operatorname{rk}E_i, \widehat{\deg}(\overline{E}_i))$.

For details about Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, see **[4, 5, 10]**.

2.3. Reminder on Borel measures. — Denote by $C_c(\mathbb{R})$ the space of all continuous functions of compact support on $\mathbb{R}$. Recall that a Borel measure on $\mathbb{R}$ is just a positive linear functional on $C_c(\mathbb{R})$, where the word "positive" means that the linear functional sends a positive function to a positive number. One says that a sequence $(\nu_n)_{n\ge1}$ of Borel measures on $\mathbb{R}$ converges *vaguely* to the Borel measure ν if, for any

$h \in C_c(\mathbb{R})$, the sequence of integrals $\big(\int h\,\mathrm{d}\nu_n\big)_{n\geq 1}$ converges to $\int h\,\mathrm{d}\nu$. This is also equivalent to the convergence of integrals for any h in $C_0^\infty(\mathbb{R})$, the space of all smooth functions of compact support on $\mathbb{R}$.

Let ν be a Borel probability measure on $\mathbb{R}$. If $a \in \mathbb{R}$, we denote by $\tau_a\nu$ the Borel measure such that $\int h\,\mathrm{d}\tau_a\nu = \int h(x+a)\nu(\mathrm{d}x)$. If $\varepsilon > 0$, let $T_\varepsilon\nu$ be the Borel measure such that $\int h\,\mathrm{d}T_\varepsilon\nu = \int h(\varepsilon x)\nu(\mathrm{d}x)$.

If ν is a Borel probability measure on $\mathbb{R}$ whose support is bounded from above, we denote by $P(\nu)$ the Legendre transformation (see **[17]** II §2.2) of the function $x \mapsto -\int_x^{+\infty}\nu(]y,+\infty[)\,\mathrm{d}y$. It is a concave function on $[0,1[$ which takes value 0 at the origin. If ν is a linear combination of Dirac measures, then $P(\nu)$ is a *polygon* (that is to say, concave and piecewise linear). An alternative definition of $P(\nu)$ is, if we denote by $F_\nu^*(t) = \sup\{x \,|\, \nu(]x,+\infty[) > t\}$, then $P(\nu)(t) = \int_0^t F_\nu^*(s)\,\mathrm{d}s$. One has $P(\tau_a\nu)(t) = P(\nu)(t) + at$ and $P(T_\varepsilon\nu) = \varepsilon P(\nu)$. For details, see **[11]** §1.2.5.

If ν_1 and ν_2 are two Borel probability measures on $\mathbb{R}$, we use the symbol $\nu_1 \succ \nu_2$ or $\nu_2 \prec \nu_1$ to denote the following condition:

for any increasing and bounded function h, $\int h\,\mathrm{d}\nu_1 \geq \int h\,\mathrm{d}\nu_2$.

It defines an order on the set of all Borel probability measures on $\mathbb{R}$. If in addition ν_1 and ν_2 are of support bounded from above, then $P(\nu_1) \geq P(\nu_2)$.

2.4. Filtered spaces. — Let k be a field and V be a vector space of finite rank over k. We call *filtration* of V any family $\mathcal{F} = (\mathcal{F}_aV)_{a\in\mathbb{R}}$ of subspaces of V subject to the following conditions

1) for all $a, b \in \mathbb{R}$ such that $a \leq b$, one has $\mathcal{F}_aV \supset \mathcal{F}_bV$,
2) $\mathcal{F}_aV = 0$ for a sufficiently positive,
3) $\mathcal{F}_aV = V$ for a sufficiently negative,
4) the function $a \mapsto \mathrm{rk}_k(\mathcal{F}_aV)$ is left continuous.

Such filtration corresponds to a flag

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_n = 0$$

together with a strictly increasing real sequence $(a_i)_{0\leq i\leq n-1}$ describing the points where the function $a \mapsto \mathrm{rk}_k(\mathcal{F}_aV)$ is discontinuous.

We define a function $\lambda : V \to \mathbb{R}\cup\{+\infty\}$ as follows:

$$\lambda(x) = \sup\{a \in \mathbb{R} \,|\, x \in \mathcal{F}_aV\}.$$

This function actually takes values in $\{a_0, \cdots, a_{n-1}, +\infty\}$, and is finite on $V \setminus \{0\}$.

If V is non-zero, the filtered space $(V, \mathcal{F})$ defines a Borel probability measure ν_V which is a linear combination of Dirac measures:

$$\nu_V = \sum_{i=0}^{n-1} \frac{\mathrm{rk}\,V_i - \mathrm{rk}\,V_{i+1}}{\mathrm{rk}\,V}\delta_{a_i}.$$

Note that the support of ν_V is just $\{a_0, \cdots, a_{n-1}\}$. We define $\lambda_{\min}(V) = a_0$ and $\lambda_{\max}(V) = a_{n-1}$. Denote by P_V the polygon $P(\nu_V)$. If $V = 0$, by convention we define ν_V as the zero measure.

If $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is an exact sequence of filtered vector spaces, where $V \neq 0$, then the following equality holds (cf. **[11]** Proposition 1.2.5):

$$\nu_V = \frac{\operatorname{rk} V'}{\operatorname{rk} V}\nu_{V'} + \frac{\operatorname{rk} V''}{\operatorname{rk} V}\nu_{V''}. \tag{6}$$

If $\overline{E}$ is a non-zero Hermitian vector bundle on $\operatorname{Spec} \mathcal{O}_K$, then the Harder-Narasimhan flag (4) and the successive slope (5) defines a filtration of $V = E_K$, called the *Harder-Narasimhan filtration.* We denote by $\nu_{\overline{E}}$ the Borel measure associated to this filtration, called the *measure associated* to the Hermitian vector bundle $\overline{E}$. One has the following relations:

$$\lambda_{\max}(V) = \widehat{\mu}_{\max}(\overline{E}), \quad \lambda_{\min}(V) = \widehat{\mu}_{\min}(\overline{E}), \quad P_V = P_{\overline{E}} = P(\nu_{\overline{E}}). \tag{7}$$

For details about filtered spaces and their measures and polygons, see **[11]** §1.2.

2.5. Slope inequality and its measure form. — For any maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, denote by $K_{\mathfrak{p}}$ the completion of K with respect to the $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$ on K, and by $|\cdot|_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic absolute value such that $|a|_{\mathfrak{p}} = \#(\mathcal{O}_K/\mathfrak{p})^{-v_{\mathfrak{p}}(a)}$.

Let $\overline{E}$ and $\overline{F}$ be two Hermitian vector bundles on $\operatorname{Spec} \mathcal{O}_K$. Let $\varphi : E_K \to F_K$ be a non-zero K-linear homomorphism. For any maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, let $\|\varphi\|_{\mathfrak{p}}$ be the norm of the linear mapping $\varphi_{K_{\mathfrak{p}}} : E_{K_{\mathfrak{p}}} \to F_{K_{\mathfrak{p}}}$. Similarly, for any embedding $\sigma : K \to \mathbb{C}$, let $\|\varphi\|_\sigma$ be the norm of $\varphi_{\sigma,\mathbb{C}} : E_{\sigma,\mathbb{C}} \to F_{\sigma,\mathbb{C}}$. The *height* of φ is then defined as

$$h(\varphi) := \sum_{\mathfrak{p}} \log \|\varphi\|_{\mathfrak{p}} + \sum_{\sigma : K \to \mathbb{C}} \log \|\varphi\|_\sigma. \tag{8}$$

Recall the *slope inequality* as follows (cf. **[4]** Proposition 4.3):

Proposition 2.1. — *If φ is injective, then $\widehat{\mu}_{\max}(\overline{E}) \leq \widehat{\mu}_{\max}(\overline{F}) + h(\varphi)$.*

The following estimation generalizing **[11]** Corollary 2.2.6 is an application of the slope inequality.

Proposition 2.2. — *Assume φ is injective. Let $\theta = \operatorname{rk} E/\operatorname{rk} F$. Then one has $\nu_{\overline{F}} \succ \theta\tau_{h(\varphi)}\nu_{\overline{E}} + (1-\theta)\delta_{\widehat{\mu}_{\min}(\overline{F})}$.*

Proof. — We equip E_K and F_K with Harder-Narasimhan filtrations. The slope inequality implies that $\lambda(\varphi(x)) \geq \lambda(x) - h(\varphi)$ for any $x \in E_K$ (see **[11]** Proposition 2.2.4). Let V be the image of φ, equipped with induced filtration. By **[11]** Corollary 2.2.6, $\nu_V \succ \tau_{h(\varphi)}\nu_{\overline{E}}$. By (6), $\nu_{\overline{F}} \succ \theta\nu_V + (1-\theta)\delta_{\widehat{\mu}_{\min}(\overline{F})}$, so the proposition is proved. □

3. Positive degree and number of effective elements

Let $\overline{E}$ be a Hermitian vector bundle on $\operatorname{Spec} \mathcal{O}_K$. Define

$$\widehat{h}^0(\overline{E}) := \log \#\{s \in E \mid \forall \sigma : K \to \mathbb{C},\ \|s\|_\sigma \leq 1\},$$

which is the logarithm of the number of effective elements in E. Note that if $0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$ is a short exact sequence of Hermitian vector bundles, then $\widehat{h}^0(\overline{E}') \le \widehat{h}^0(\overline{E})$. If in addition $\widehat{h}^0(\overline{E}'') = 0$, then $\widehat{h}^0(\overline{E}') = \widehat{h}^0(\overline{E})$.

In this section, we define an invariant of $\overline{E}$, suggested by J.-B. Bost, which is called the *positive degree*:

$$\widehat{\deg}_+(\overline{E}) := \max_{t\in[0,1]} P_{\overline{E}}(t).$$

If E is non-zero, define the *positive slope* of $\overline{E}$ as $\widehat{\mu}_+(\overline{E}) = \widehat{\deg}_+(\overline{E})/\operatorname{rk} E$. Using the Riemann-Roch inequality established by Gillet and Soulé **[15]**, we shall compare $\widehat{h}^0(\overline{E})$ to $\widehat{\deg}_+(\overline{E})$.

3.1. Reminder on dualizing bundle and Riemann-Roch inequality. — Denote by $\overline{\omega}_{\mathcal{O}_K}$ the *arithmetic dualizing bundle* on $\operatorname{Spec}\mathcal{O}_K$: it is a Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$ whose underlying $\mathcal{O}_K$-module is $\omega_{\mathcal{O}_K} := \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K,\mathbb{Z})$. This $\mathcal{O}_K$-module is generated by the trace map $\operatorname{tr}_{K/\mathbb{Q}} : K \to \mathbb{Q}$ up to torsion. We choose Hermitian metrics on $\omega_{\mathcal{O}_K}$ such that $\|\operatorname{tr}_{K/\mathbb{Q}}\|_\sigma = 1$ for any embedding $\sigma : K \to \mathbb{C}$. The arithmetic degree of $\overline{\omega}_{\mathcal{O}_K}$ is $\log|\Delta_K|$, where Δ_K is the discriminant of K over $\mathbb{Q}$.

We recall below a result in **[15]**, which should be considered as an arithmetic analogue of classical Riemann-Roch formula for vector bundles on a smooth projective curve.

***Proposition 3.1* (Gillet and Soulé).** — *There exists an increasing function $C_0 : \mathbb{N}_* \to \mathbb{R}_+$ satisfying $C_0(n) \ll_K n\log n$ such that, for any Hermitian vector bundle $\overline{E}$ on* $\operatorname{Spec}\mathcal{O}_K$*, one has*

$$\big|\widehat{h}^0(\overline{E}) - \widehat{h}^0(\overline{\omega}_{\mathcal{O}_K}\otimes\overline{E}^\vee) - \widehat{\deg}(\overline{E})\big| \le C_0(\operatorname{rk} E). \tag{9}$$

3.2. Comparison of $\widehat{h}^0$ and $\widehat{\deg}_+$. — Proposition 3.3 below is a comparison between $\widehat{h}^0$ and $\widehat{\deg}_+$. The following lemma, which is consequences of the Riemann-Roch inequality (9), is needed for the proof.

Lemma 3.2. — *Let $\overline{E}$ be a non-zero Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$*.*

1) If $\widehat{\mu}_{\max}(\overline{E}) < 0$, then $\widehat{h}^0(\overline{E}) = 0$.

2) If $\widehat{\mu}_{\min}(\overline{E}) > \log|\Delta_K|$, then $\big|\widehat{h}^0(\overline{E}) - \widehat{\deg}(\overline{E})\big| \le C_0(\operatorname{rk} E)$.

3) If $\widehat{\mu}_{\min}(\overline{E}) \ge 0$, then $\big|\widehat{h}^0(\overline{E}) - \widehat{\deg}(\overline{E})\big| \le \log|\Delta_K|\operatorname{rk} E + C_0(\operatorname{rk} E)$.

Proof. — 1) Assume that $\overline{E}$ has an effective section. There then exists a homomorphism $\varphi : \overline{\mathscr{L}}_0 \to \overline{E}$ whose height is negative or zero. By slope inequality, we obtain $\widehat{\mu}_{\max}(\overline{E}) \ge 0$.

2) Since $\widehat{\mu}_{\min}(\overline{E}) > \log|\Delta_K|$, we have $\widehat{\mu}_{\max}(\overline{\omega}_{\mathcal{O}_K}\otimes\overline{E}^\vee) < 0$. By 1), $\widehat{h}^0(\overline{\omega}_{\mathcal{O}_K}\otimes\overline{E}^\vee) = 0$. Thus the desired inequality results from (9).

3) Let $a = \log|\Delta_K| + \varepsilon$ with $\varepsilon > 0$. Then $\widehat{\mu}_{\min}(\overline{E}\otimes\overline{\mathscr{L}}_a) > \log|\Delta_K|$. By 2), $\widehat{h}^0(\overline{E}\otimes\overline{\mathscr{L}}_a) \le \widehat{\deg}(\overline{E}\otimes\overline{\mathscr{L}}_a) + C_0(\operatorname{rk} E) = \widehat{\deg}(\overline{E}) + a\operatorname{rk} E + C_0(\operatorname{rk} E)$. Since $a > 0$, $\widehat{h}^0(\overline{E}) \le \widehat{h}^0(\overline{E}\otimes\overline{\mathscr{L}}_a)$. So we obtain $\widehat{h}^0(\overline{E}) - \widehat{\deg}(\overline{E}) \le a\operatorname{rk} E + C_0(\operatorname{rk} E)$. Moreover, (9) implies $\widehat{h}^0(\overline{E}) - \widehat{\deg}(\overline{E}) \ge \widehat{h}^0(\overline{\omega}_{\mathcal{O}_K}\otimes\overline{E}^\vee) - C_0(\operatorname{rk} E) \ge -C_0(\operatorname{rk} E)$. Therefore, we

always have $\big|\widehat{h}^0(\overline{E})-\widehat{\deg}(\overline{E})\big|\le a\operatorname{rk}E+C_0(\operatorname{rk}E)$. Since ε is arbitrary, we obtain the desired inequality. □

Proposition 3.3. — *The following inequality holds:*

$$\big|\widehat{h}^0(\overline{E})-\widehat{\deg}_+(\overline{E})\big|\le\operatorname{rk}E\log|\Delta_K|+C_0(\operatorname{rk}E). \tag{10}$$

Proof. — Let the Harder-Narasimhan flag of $\overline{E}$ be as in (4). For any integer i such that $0\le i\le n-1$, let $\alpha_i=\widehat{\mu}(\overline{E}_i/\overline{E}_{i+1})$. Let j be the first index in $\{0,\cdots,n-1\}$ such that $\alpha_j\ge 0$; if such index does not exist, let $j=n$. By definition, $\widehat{\deg}_+(\overline{E})=\widehat{\deg}(\overline{E}_j)$. Note that, if $j>0$, then $\widehat{\mu}_{\max}(\overline{E}/\overline{E}_j)=\alpha_{j-1}<0$. Therefore we always have $h^0(\overline{E}/\overline{E}_j)=0$ and hence $\widehat{h}^0(\overline{E})=\widehat{h}^0(\overline{E}_j)$.

If $j=n$, then $\widehat{h}^0(\overline{E}_j)=0=\widehat{\deg}_+(\overline{E})$. Otherwise $\widehat{\mu}_{\min}(\overline{E}_j)=\alpha_j\ge 0$ and by Lemma 3.2 3), we obtain

$$\begin{aligned}&\big|\widehat{h}^0(\overline{E})-\widehat{\deg}_+(\overline{E})\big|=\big|\widehat{h}^0(\overline{E}_j)-\widehat{\deg}(\overline{E}_j)\big|\\ &\le\operatorname{rk}E_j\log|\Delta_K|+C_0(\operatorname{rk}E_j)\le\operatorname{rk}E\log|\Delta_K|+C_0(\operatorname{rk}E).\end{aligned}$$

□

4. Asymptotic polygon of a big line bundle

Let k be a field and $B=\bigoplus_{n\ge 0}B_n$ be an integral graded k-algebra such that, for n sufficiently positive, B_n is non-zero and has finite rank. Let $f:\mathbb{N}^*\to\mathbb{R}_+$ be a mapping such that $\lim\limits_{n\to\infty}f(n)/n=0$. Assume that each vector space B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}^{(n)}$ such that B is f*-quasi-filtered* (cf. **[11]** §3.2.1). In other words, we assume that there exists $n_0\in\mathbb{N}^*$ such that, for any integer $r\ge 2$ and all homogeneous elements $x_1,\cdots,x_r$ in B respectively of degree $n_1,\cdots,n_r$ in $\mathbb{N}_{\ge n_0}$, one has

$$\lambda(x_1\cdots x_r)\ge\sum_{i=1}^r\Big(\lambda(x_i)-f(n_i)\Big).$$

We suppose in addition that $\sup_{n\ge 1}\lambda_{\max}(B_n)/n<+\infty$. Recall below some results in **[11]** (Proposition 3.2.4 and Theorem 3.4.3).

Proposition 4.1. — *1) The sequence $(\lambda_{\max}(B_n)/n)_{n\ge 1}$ converges in $\mathbb{R}$.*
2) If B is finitely generated, then the sequence of measures $(T_{\frac{1}{n}}\nu_{B_n})_{n\ge 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$.

In this section, we shall generalize the second assertion of Proposition 4.1 to the case where the algebra B is given by global sections of tensor power of a big line bundle on a projective variety.

4.1. Convergence of measures. — Let X be an integral projective scheme of dimension d defined over k and L be a *big* invertible $\mathcal{O}_X$-module: recall that an invertible $\mathcal{O}_X$-module L is said to be big if its *volume*, defined as

$$\mathrm{vol}(L) := \limsup_{n\to\infty} \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^d/d!},$$

is strictly positive.

Theorem 4.2. — *With the above notation, if* $B = \bigoplus_{n\geq 0} H^0(X, L^{\otimes n})$*, then the sequence of measures* $(T_{\frac{1}{n}}\nu_{B_n})_{n\geq 1}$ *converges vaguely to a probability measure on* $\mathbb{R}$.

Proof. — For any integer $n \geq 1$, denote by ν_n the measure $T_{\frac{1}{n}}\nu_{B_n}$. Since L is big, for sufficiently positive n, $H^0(X, L^{\otimes n}) \neq 0$, and hence ν_n is a probability measure. In addition, Proposition 4.1 1) implies that the supports of the measures ν_n are uniformly bounded from above. After Fujita's approximation theorem (cf. **[13, 23]**, see also **[18]** Ch. 11), the volume function $\mathrm{vol}(L)$ is in fact a limit:

$$\mathrm{vol}(L) = \lim_{n\to\infty} \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^d/d!}.$$

Furthermore, for any real number ε, $0 < \varepsilon < 1$, there exists an integer $p \geq 1$ together with a finitely generated sub-k-algebra A^ε of $B^{(p)} = \bigoplus_{n\geq 0} B_{np}$ which is generated by elements in B_p and such that

$$\lim_{n\to\infty} \frac{\mathrm{rk}(B_{np}) - \mathrm{rk}(A^\varepsilon_n)}{\mathrm{rk}(B_{np})} \leq \varepsilon.$$

The graded k-algebra A^ε, equipped with induced filtrations, is f-quasi-filtered. Therefore Proposition 4.1 2) is valid for A^ε. In other words, If we denote by ν^ε_n the Borel measure $T_{\frac{1}{np}}\nu_{A^\varepsilon_n} = T_{\frac{1}{p}}(T_{\frac{1}{n}}\nu_{A^\varepsilon_n})$, then the sequence of measures $(\nu^\varepsilon_n)_{n\geq 1}$ converges vaguely to a Borel probability measure ν^ε on $\mathbb{R}$. In particular, for any function $h \in C_c(\mathbb{R})$, the sequence of integrals $\big(\int h\,\mathrm{d}\nu^\varepsilon_n\big)_{n\geq 1}$ is a Cauchy sequence. This assertion is also true when h is a continuous function on $\mathbb{R}$ whose support is bounded from below: the supports of the measures ν^ε_n are uniformly bounded from above. The exact sequence $0 \longrightarrow A^\varepsilon_n \longrightarrow B_{np} \longrightarrow B_{np}/A^\varepsilon_n \longrightarrow 0$ implies that

$$\nu_{B_{np}} = \frac{\mathrm{rk}\, A^\varepsilon_n}{\mathrm{rk}\, B_{np}}\nu_{A^\varepsilon_n} + \frac{\mathrm{rk}\, B_{np} - \mathrm{rk}\, A^\varepsilon_n}{\mathrm{rk}\, B_{np}}\nu_{B_{np}/A^\varepsilon_n}.$$

Therefore, for any bounded Borel function h, one has

$$\Big|\int h\,\mathrm{d}\nu_{np} - \frac{\mathrm{rk}\, A^\varepsilon_n}{\mathrm{rk}\, B_{np}}\int h\,\mathrm{d}\nu^\varepsilon_n\Big| \leq \|h\|_{\sup}\frac{\mathrm{rk}\, B_{np} - \mathrm{rk}\, A^\varepsilon_n}{\mathrm{rk}\, B_{np}}. \tag{11}$$

Hence, for any bounded continuous function h satisfying $\inf(\mathrm{supp}(h)) > -\infty$, there exists $N_{h,\varepsilon} \in \mathbb{N}$ such that, for any $n, m \geq N_{h,\varepsilon}$,

$$\Big|\int h\,\mathrm{d}\nu_{np} - \int h\,\mathrm{d}\nu_{mp}\Big| \leq 2\varepsilon\|h\|_{\sup} + \varepsilon. \tag{12}$$

Let h be a smooth function on $\mathbb{R}$ whose support is compact. We choose two increasing continuous functions h_1 and h_2 such that $h = h_1 - h_2$ and that the supports of them are bounded from below. Let $n_0 \in \mathbb{N}^*$ be sufficiently large such that, for any $r \in \{n_0p+1, \cdots, n_0p+p-1\}$, one has $H^0(X, L^{\otimes r}) \neq 0$. We choose, for such r, a non-zero element $e_r \in H^0(X, L^{\otimes r})$. For any $n \in \mathbb{N}$ and any $r \in \{n_0p+1, \cdots, n_0p+p-1\}$, let $M_{n,r} = e_r B_{np} \subset B_{np+r}$, $M'_{n,r} = e_{2n_0p+p-r} M_{n,r} \subset B_{(n+2n_0+1)p}$ and denote by $\nu_{n,r} = T_{\frac{1}{np}} \nu_{M_{n,r}}$, $\nu'_{n,r} = T_{\frac{1}{np}} \nu_{M'_{n,r}}$, where $M_{n,r}$ and $M'_{n,r}$ are equipped with the induced filtrations. As the algebra B is f-quasi-filtered, we obtain, by **[11]** Lemma 1.2.6, $\nu'_{n,r} \succ \tau_{a_{n,r}} \nu_{n,r} \succ \tau_{b_{n,r}} \nu_{np}$, where

$$a_{n,r} = \frac{\lambda(e_{2n_0p+p-r}) - f(np+r) - f(2n_0p+p-r)}{np}, \; b_{n,r} = a_{n,r} + \frac{\lambda(e_r) - f(np) - f(r)}{np}.$$

This implies

$$\int h_i \,\mathrm{d}\nu'_{n,r} \geq \int h_i \,\mathrm{d}\tau_{a_{n,r}} \nu_{n,r} \geq \int h_i \,\mathrm{d}\tau_{b_{n,r}} \nu_{np}, \quad i = 1, 2. \tag{13}$$

In particular,

$$\Big| \int h_i \,\mathrm{d}\tau_{a_{n,r}} \nu_{n,r} - \int h_i \,\mathrm{d}\tau_{b_{n,r}} \nu_{np} \Big| \leq \Big| \int h_i \,\mathrm{d}\nu'_{n,r} - \int h_i \,\mathrm{d}\tau_{b_{n,r}} \nu_{np} \Big| \tag{14}$$

As $\displaystyle\lim_{n\to\infty} \frac{\operatorname{rk} B_{(n+2n_0+1)p} - \operatorname{rk} B_{np}}{\operatorname{rk} B_{(n+2n_0+1)p}} = 0$,

$$\lim_{n\to\infty} \Big| \int h_i \,\mathrm{d}\nu'_{n,r} - \int h_i \,\mathrm{d}\nu_{(n+2n_0+1)p} \Big| = 0. \tag{15}$$

Moreover, $\lim_{n\to\infty} b_{n,r} = 0$. By **[11]** Lemma 1.2.10, we obtain

$$\lim_{n\to\infty} \Big| \int h_i \,\mathrm{d}\tau_{b_{n,r}} \nu_{np} - \int h_i \,\mathrm{d}\nu_{np} \Big| = 0. \tag{16}$$

We deduce, from (12), (15) and (16),

$$\begin{aligned} &\limsup_{n\to\infty} \Big| \int h_i \,\mathrm{d}\nu'_{n,r} - \int h_i \,\mathrm{d}\tau_{b_{n,r}} \nu_{np} \Big| \\ &= \limsup_{n\to\infty} \Big| \int h_i \,\mathrm{d}\nu_{(n+2n_0+1)p} - \int h_i \,\mathrm{d}\nu_{np} \Big| \leq 2\varepsilon \|h_i\|_{\sup} + \varepsilon. \end{aligned} \tag{17}$$

By (14),

$$\limsup_{n\to\infty} \Big| \int h_i \,\mathrm{d}\tau_{a_{n,r}} \nu_{n,r} - \int h_i \,\mathrm{d}\tau_{b_{n,r}} \nu_{np} \Big| \leq 2\varepsilon \|h_i\|_{\sup} + \varepsilon. \tag{18}$$

Note that

$$\lim_{n\to\infty} \frac{\operatorname{rk} B_{np+r} - \operatorname{rk} B_{np}}{\operatorname{rk} B_{np+r}} = \lim_{n\to\infty} a_{n,r} = 0.$$

So

$$\lim_{n\to\infty} \Big| \int h_i \,\mathrm{d}\nu_{n,r} - \int h_i \,\mathrm{d}\nu_{np+r} \Big| = \lim_{n\to\infty} \Big| \int h_i \,\mathrm{d}\nu_{n,r} - \int h_i \,\mathrm{d}\tau_{a_{n,r}} \nu_{n,r} \Big| = 0.$$

Hence, by (16) and (18), we have

$$\limsup_{n\to\infty} \Big| \int h \,\mathrm{d}\nu_{np+r} - \int h \,\mathrm{d}\nu_{np} \Big| \leq 2\varepsilon (\|h_1\|_{\sup} + \|h_2\|_{\sup}) + 2\varepsilon.$$

According to (12), we obtain that there exists $N'_{h,\varepsilon} \in \mathbb{N}^*$ such that, for all integers l and l' such that $l \geq N'_{h,\varepsilon}$, $l' \geq N'_{h,\varepsilon}$, one has

$$\Big|\int h\,\mathrm{d}\nu_l - \int h\,\mathrm{d}\nu_{l'}\Big| \leq 4\varepsilon(\|h_1\|_{\sup} + \|h_2\|_{\sup}) + 2\varepsilon\|h\|_{\sup} + 6\varepsilon,$$

which implies that the sequence $(\int h\,\mathrm{d}\nu_n)_{n\geq 1}$ converges in $\mathbb{R}$.

Let $I : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ be the linear functional defined by $I(h) = \lim\limits_{n\to\infty}\int h\,\mathrm{d}\nu_n$. It extends in a unique way to a continuous linear functional on $C_c(\mathbb{R})$. Furthermore, it is positive, and so defines a Borel measure ν on $\mathbb{R}$. Finally, by (11), $|\nu(\mathbb{R}) - (1-\varepsilon)\nu^\varepsilon(\mathbb{R})| \leq \varepsilon$. Since ε is arbitrary, ν is a probability measure. □

4.2. Convergence of maximal values of polygons. — If ν is a Borel probability measure on $\mathbb{R}$ and $\alpha \in \mathbb{R}$, denote by $\nu^{(\alpha)}$ the Borel probability measure on $\mathbb{R}$ such that, for any $h \in C_c(\mathbb{R})$,

$$\int h\,\mathrm{d}\nu^{(\alpha)} = \int h(x)1\!\!1_{[\alpha,+\infty[}(x)\nu(dx) + h(\alpha)\nu(]-\infty,\alpha[).$$

The measure $\nu^{(\alpha)}$ is called the *truncation* of ν at α. The truncation operator preserves the order "$\succ$".

Assume that the support of ν is bounded from above. The truncation of ν at α modifies the "polygon" $P(\nu)$ only on the part with derivative $< \alpha$. More precisely, one has

$$P(\nu) = P(\nu^{(\alpha)}) \text{ on } \{t \in [0,1[\,\big|\, F_\nu^*(t) \geq \alpha\}.$$

In particular, if $\alpha \leq 0$, then

$$\max_{t\in[0,1[} P(\nu)(t) = \max_{t\in[0,1[} P(\nu^{(\alpha)})(t). \tag{19}$$

The following proposition shows that given a vague convergence sequence of Borel probability measures, almost all truncations preserve vague limit.

Proposition 4.3. — *Let $(\nu_n)_{n\geq 1}$ be a sequence of Borel probability measures which converges vaguely to a Borel probability measure ν. Then, for any $\alpha \in \mathbb{R}$, the sequence $(\nu_n^{(\alpha)})_{n\geq 1}$ converges vaguely to $\nu^{(\alpha)}$.*

Proof. — For any $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, write $x\vee\alpha := \max\{x,\alpha\}$. Assume that $h \in C_c(\mathbb{R})$ and η is a Borel probability measure. Then

$$\int h(x)\eta^{(\alpha)}(\mathrm{d}x) = \int h(x\vee\alpha)\eta(\mathrm{d}x).$$

By **[9]** IV §5 n°12 Proposition 22, for any $\alpha \in \mathbb{R}$,

$$\lim_{n\to\infty}\int h(x)\nu_n^{(\alpha)}(\mathrm{d}x) = \lim_{n\to\infty}\int h(x\vee\alpha)\nu_n(\mathrm{d}x) = \int h(x\vee\alpha)\nu(\mathrm{d}x) = \int h(x)\nu^{(\alpha)}(\mathrm{d}x).$$

Therefore, the proposition is proved. □

Corollary 4.4. — *Under the assumption of Theorem 4.2, the sequence*

$$\big(\max_{t\in[0,1]} P_{B_n}(t)/n\big)_{n\geq 1}$$

converges in $\mathbb{R}$.

Proof. — For $n\in\mathbb{N}^*$, denote by $\nu_n = T_{\frac{1}{n}}\nu_{B_n}$. By Theorem 4.2, the sequence $(\nu_n)_{n\geq 1}$ converges vaguely to a Borel probability measure ν. Let $\alpha<0$ be a number such that $(\nu_n^{(\alpha)})_{n\geq 1}$ converges vaguely to $\nu^{(\alpha)}$. Note that the supports of ν_n^{α} are uniformly bounded. So $P(\nu_n^{(\alpha)})$ converges uniformly to $P(\nu^{(\alpha)})$ (see **[11]** Proposition 1.2.9). By (19), $\big(\max_{t\in[0,1]} P_{B_n}(t)/n\big)_{n\geq 1}$ converges to $\max_{t\in[0,1]} P(\nu)(t)$. □

If V is a finite dimensional filtered vector space over k, we shall use the expression $\lambda_+(V)$ to denote $\max_{t\in[0,1]} P_V(t)$. With this notation, the assertion of Corollary 4.4 becomes: $\lim_{n\to\infty}\lambda_+(B_n)/n$ exists in $\mathbb{R}$.

Lemma 4.5. — *Assume that* ν_1 *and* ν_2 *are two Borel probability measures whose supports are bounded from above. Let* $\varepsilon\in]0,1[$ *and* $\nu=\varepsilon\nu_1+(1-\varepsilon)\nu_2$. *Then*

$$\max_{t\in[0,1]} P(\nu)(t)\geq\varepsilon\max_{t\in[0,1]} P(\nu_1)(t). \tag{20}$$

Proof. — After truncation at 0 we may assume that the supports of ν_1 and ν_2 are contained in $[0,+\infty[$. In this case $\nu\succ\varepsilon\nu_1+(1-\varepsilon)\delta_0$ and hence $P(\nu)\geq P(\varepsilon\nu_1+(1-\varepsilon)\delta_0)$. Since

$$P(\varepsilon\nu_1+(1-\varepsilon)\delta_0)(t)=\begin{cases}\varepsilon P(\nu_1)(t/\varepsilon), & t\in[0,\varepsilon],\\ \varepsilon P(\nu_1)(1), & t\in[\varepsilon,1[,\end{cases}$$

we obtain (20). □

Theorem 4.6. — *Under the assumption of Theorem 4.2, one has*

$$\lim_{n\to\infty}\lambda_+(B_n)/n>0 \quad \textit{if and only if} \quad \lim_{n\to\infty}\lambda_{\max}(B_n)/n>0.$$

Furthermore, in this case, the inequality $\lim_{n\to\infty}\lambda_+(B_n)/n\leq\lim_{n\to\infty}\lambda_{\max}(B_n)/n$ *holds.*

Proof. — For any filtered vector space V, $\lambda_{\max}(V)>0$ if and only if $\lambda_+(V)>0$, and in this case one always has $\lambda_{\max}(V)\geq\lambda_+(V)$. Therefore the second assertion is true. Furthermore, this also implies

$$\lim_{n\to\infty}\frac{1}{n}\lambda_+(B_n)>0\Longrightarrow\lim_{n\to\infty}\frac{1}{n}\lambda_{\max}(B_n)>0.$$

It suffices then to prove the converse implication. Assume that $\alpha>0$ is a real number such that $\lim_{n\to\infty}\lambda_{\max}(B_n)/n>4\alpha$. Choose sufficiently large $n_0\in\mathbb{N}$ such that $f(n)<\alpha n$ for any $n\geq n_0$ and such that there exists a non-zero $x_0\in B_{n_0}$ satisfying $\lambda(x_0)\geq 4\alpha n_0$. Since the algebra B is f-quasi-filtered, $\lambda(x_0^m)\geq 4\alpha n_0 m-mf(n)\geq 3\alpha m n_0$. By Fujita's approximation theorem, there exists an integer p divisible by n_0 and a subalgebra A of $B^{(p)}=\bigoplus_{n\geq 0}B_{np}$ generated by a finite number of elements in B_p and such that $\liminf_{n\to\infty}\operatorname{rk}A_n/\operatorname{rk}B_{np}>0$. By possible enlargement of A we may assume that A contains x_0^{p/n_0}. By Lemma 4.5, $\lim_{n\to\infty}\lambda_+(A_n)/n > 0$ implies $\lim_{n\to\infty}\lambda_+(B_{np})/np = \lim_{n\to\infty}\lambda_+(B_n)/n > 0$. Therefore, we reduce our problem to the case where

1) B is an algebra of finite type generated by B_1,
2) there exists $x_1 \in B_1$, $x_1 \neq 0$ such that $\lambda(x_1) \geq 3\alpha$ with $\alpha > 0$,
3) $f(n) < \alpha n$.

Furthermore, by Noether's normalization theorem, we may assume that $B = k[x_1, \cdots, x_q]$ is an algebra of polynomials, where x_1 coincides with the element in condition 2). Note that

$$\lambda(x_1^{a_1}\cdots x_q^{a_q}) \geq \sum_{i=1}^{q} a_i\big(\lambda(x_i)-\alpha\big) \geq 2\alpha a_1 + \sum_{i=2}^{q} a_i\big(\lambda(x_i)-\alpha\big). \tag{21}$$

Let $\beta > 0$ such that $-\beta \leq \lambda(x_i) - \alpha$ for any $i \in \{2, \cdots, q\}$. We obtain from (21) that $\lambda(x_1^{a_1}\cdots x_q^{a_q}) \geq \alpha a_1$ as soon as $a_1 \geq \frac{\beta}{\alpha}\sum_{i=2}^{q} a_i$. For $n \in \mathbb{N}^*$, let

$$\begin{aligned}
u_n &= \#\Big\{(a_1, \cdots, a_q) \in \mathbb{N}^q \,\Big|\, a_1 + \cdots + a_q = n,\ a_1 \geq \frac{\beta}{\alpha}(a_2 + \cdots + a_q)\Big\} \\
&= \#\Big\{(a_1, \cdots, a_q) \in \mathbb{N}^q \,\Big|\, a_1 + \cdots + a_q = n,\ a_1 \geq \frac{\beta}{\alpha+\beta}n\Big\} \\
&= \binom{n - \lfloor\frac{\beta}{\alpha+\beta}n\rfloor + q - 1}{q-1},
\end{aligned}$$

and

$$v_n = \#\big\{(a_1, \cdots, a_q) \in \mathbb{N}^q \,\big|\, a_1 + \cdots + a_q = n\big\} = \binom{n+q-1}{q-1}.$$

Thus $\lim_{n\to\infty} u_n/v_n = \Big(\dfrac{\alpha}{\alpha+\beta}\Big)^{q-1} > 0$, which implies $\lim_{n\to\infty}\dfrac{1}{n}\lambda_+(B_n) > 0$ by Lemma 4.5. □

5. Volume function as a limit and arithmetic bigness

Let $\mathcal{X}$ be an arithmetic variety of dimension d and $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$. Denote by $X = \mathcal{X}_K$ and $L = \mathcal{L}_K$. Using the convergence result established in the previous section, we shall prove that the volume function is in fact a limit of normalized positive degrees. We also give a criterion of arithmetic bigness by the positivity of asymptotic maximal slope.

5.1. Volume function and asymptotic positive degree. — For any $n \in \mathbb{N}$, we choose a Hermitian vector bundle $\pi_*(\overline{\mathcal{L}}^{\otimes n}) = (\pi_*(\mathcal{L}^{\otimes n}), (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}})$ whose underlying $\mathcal{O}_K$-module is $\pi_*(\mathcal{L}^{\otimes n})$ and such that

$$\max_{0\neq s\in\pi_*(\mathcal{L}^{\otimes n})}\Big|\log\|s\|_{\sup} - \log\|s\|_\sigma\Big| \ll \log n, \qquad n > 1. \tag{22}$$

Denote by r_n the rank of $\pi_*(\mathcal{L}^{\otimes n})$. One has $r_n \ll n^{d-1}$. For any $n\in\mathbb{N}$, define

$$\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n}):=\log\#\{s\in H^0(\mathcal{X},\mathcal{L}^{\otimes n})\mid \forall\sigma:K\rightarrow\mathbb{C},\ \|s\|_{\sigma,\sup}\leq 1\}.$$

Recall that the arithmetic volume function of $\overline{\mathcal{L}}$ defined by Moriwaki (cf. [**20**]) is

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}):=\limsup_{n\rightarrow\infty}\frac{\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})}{n^d/d!},$$

and $\overline{\mathcal{L}}$ is said to be big if and only if $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})>0$ (cf. [**20**] Theorem 4.5 and [**25**] Corollary 2.4). We emphasize that this definition does not depend on the Hermitian metrics on $\pi_*(\mathcal{L}^{\otimes n})$ satisfying the condition (22).

In the following, we give an alternative proof of a result of Morkwaki and Yuan.

Proposition 5.1. — *If $\overline{\mathcal{L}}$ is big, then L is big on X in usual sense.*

Proof. — For any integer $n\geq 1$, we choose two Hermitian vector bundles $\overline{E}_n^{(1)}=(\pi_*(\mathcal{L}^{\otimes n}),(\|\cdot\|_\sigma^{(1)})_{\sigma:K\rightarrow\mathbb{C}})$ and $\overline{E}_n^{(2)}=(\pi_*(\mathcal{L}^{\otimes n}),(\|\cdot\|_\sigma^{(2)})_{\sigma:K\rightarrow\mathbb{C}})$ such that

$$\|s\|_\sigma^{(1)}\leq\|s\|_{\sigma,\sup}\leq\|s\|_\sigma^{(2)}\leq r_n\|s\|_\sigma^{(1)},$$

where r_n is the rank of $\pi_*(\mathcal{L}^{\otimes n})$. This is always possible due to an argument of John and Löwner ellipsoid, see [**14**] definition-theorem 2.4. Thus

$$\widehat{h}^0(\overline{E}_n^{(2)})\leq\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})\leq\widehat{h}^0(\overline{E}_n^{(1)}).$$

Furthermore, by [**11**] Corollary 2.2.9, $\big|\widehat{\deg}_+(\overline{E}_n^{(1)})-\widehat{\deg}_+(\overline{E}_n^{(2)})\big|\leq r_n\log r_n$. By (10), we obtain

$$\big|\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})-\widehat{h}^0(\overline{E}_n^{(1)})\big|\leq 2r_n\log|\Delta_K|+2C_0(r_n)+r_n\log r_n.$$

Furthermore, $\big|\widehat{\deg}_+(\overline{E}_n^{(1)})-\widehat{\deg}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))\big|=O(r_n\log r_n)$. Hence

$$\big|\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})-\widehat{h}^0(\pi_*(\overline{\mathcal{L}}^{\otimes n}))\big|=O(r_n\log r_n).$$

Since $r_n\ll n^{d-1}$, we obtain

$$\lim_{n\rightarrow\infty}\left|\frac{\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})}{n^d/d!}-\frac{\widehat{\deg}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n^d/d!}\right|=0, \tag{23}$$

and therefore $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})=\limsup\limits_{n\rightarrow\infty}\dfrac{\widehat{\deg}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n^d/d!}$. If $\overline{\mathcal{L}}$ is big, then $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})>0$, and hence $\pi_*(\mathcal{L}^{\otimes n})\neq 0$ for n sufficiently positive. Combining with the fact that

$$\limsup_{n\rightarrow+\infty}\frac{\widehat{\deg}_+(\pi_*(\mathcal{L}^{\otimes n}))}{nr_n}\leq\lim_{n\rightarrow+\infty}\frac{\widehat{\mu}_{\max}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n}<+\infty,$$

we obtain $\limsup\limits_{n\rightarrow+\infty}\dfrac{r_n}{n^{d-1}}>0$. □

Theorem 5.2. — *The following equalities hold:*
(24)
$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \lim_{n\to\infty}\frac{\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})}{n^d/d!} = \lim_{n\to\infty}\frac{\widehat{\deg}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n^d/d!} = \mathrm{vol}(L)\lim_{n\to\infty}\frac{\widehat{\mu}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n/d},$$
where the positive slope $\widehat{\mu}_+$ was defined in §3.

Proof. — We first consider the case where L is big. The graded algebra $B = \bigoplus_{n\geq 0} H^0(X, L^{\otimes n})$ equipped with Harder-Narasimhan filtrations is quasi-filtered for a function of logarithmic increasing speed at infinity (see **[11]** §4.1.3). Therefore Corollary 4.4 shows that the sequence $(\lambda_+(B_n)/n)_{n\geq 1}$ converges in $\mathbb{R}$. Note that $\lambda_+(B_n) = \widehat{\mu}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))$. So the last limit in (24) exists. Furthermore, L is big on X, so
$$\mathrm{vol}(L) = \lim_{n\to\infty}\frac{\mathrm{rk}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n^{d-1}/(d-1)!},$$
which implies the existence of the third limit in (24) and the last equality. Thus the existence of the first limit and the second equality follow from (23).

When L is not big, since
$$\lim_{n\to\infty}\frac{\widehat{\mu}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n/d} \leq \lim_{n\to\infty}\frac{\widehat{\mu}_{\max}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n/d} < +\infty$$
the last term in (24) vanishes. This implies the vanishing of the second limit in (24). Also by (23), we obtain the vanishing of the first limit.

□

Corollary 5.3. — *The following relations hold:*

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) \geq \limsup_{n\to\infty}\frac{\widehat{\deg}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n^d/d!} = \limsup_{n\to\infty}\frac{\chi(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n^d/d!}. \tag{25}$$

Proof. — The inequality is a consequence of Theorem 5.2 and the comparison $\widehat{\deg}_+(\overline{E}) \geq \widehat{\deg}(\overline{E})$. Here $\overline{E}$ is an arbitrary Hermitian vector bundle on $\mathrm{Spec}\,\mathcal{O}_K$. The equality follows from a classical result which compares Arakelov degree and Euler-Poincaré characteristic. □

5.2. A criterion of arithmetic bigness. — We shall prove that the bigness of $\overline{\mathcal{L}}$ is equivalent to the positivity of the asymptotic maximal slope of $\overline{\mathcal{L}}$. This result is strongly analogous to Theorem 4.5 of **[20]**. In fact, by a result of Borek **[3]** (see also **[6]** Proposition 3.3.1), which reformulate Minkowski's First Theorem, the maximal slope of a Hermitian vector bundle on $\mathrm{Spec}\,\mathcal{O}_K$ is "close" to the opposite of the logarithm of its first minimum. So the positivity of the asymptotic maximal slope is equivalent to the existence of (exponentially) small section when n goes to infinity.

Theorem 5.4. — *$\overline{\mathcal{L}}$ is big if and only if $\lim_{n\to\infty}\widehat{\mu}_{\max}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))/n>0$. Furthermore, the following inequality holds:*

$$\frac{\widehat{\mathrm{vol}}(\overline{\mathcal{L}})}{\mathrm{dvol}(L)}\le\lim_{n\to\infty}\frac{\widehat{\mu}_{\max}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n}.$$

Proof. — Since both conditions imply the bigness of L, we may assume that L is big. Let $B=\bigoplus_{n\ge 0}H^0(X,L^{\otimes n})$ equipped with Harder-Narasimhan filtrations. One has

$$\widehat{\mu}_+(\pi_*(\overline{\mathcal{L}}^{\otimes n}))=\lambda_+(B_n),\qquad \widehat{\mu}_{\max}(\pi_*(\overline{\mathcal{L}}^{\otimes n})=\lambda_{\max}(B_n).$$

Therefore, the assertion follows from Theorems 4.6 and 5.2. □

Remark 5.5. — After **[6]** Proposition 3.3.1, for any non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$, one has

$$\Big|\widehat{\mu}_{\max}(\overline{E})+\frac{1}{2}\log\inf_{0\ne s\in E}\sum_{\sigma:K\to\mathbb{C}}\|s\|_\sigma^2\Big|\le\frac{1}{2}\log[K:\mathbb{Q}]+\frac{1}{2}\log\operatorname{rk}E+\frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}.\tag{26}$$

Therefore, by (26), the bigness of $\overline{\mathcal{L}}$ is equivalent to each of the following conditions:

1) L is big, and there exists $\varepsilon>0$ such that, when n is sufficiently large, $\overline{\mathcal{L}}^{\otimes n}$ has a global section s_n satisfying $\|s_n\|_{\sigma,\sup}\le e^{-\varepsilon n}$ for any $\sigma:K\to\mathbb{C}$.
2) L is big, and there exists an integer $n\ge 1$ such that $\overline{\mathcal{L}}^{\otimes n}$ has a global section s_n satisfying $\|s_n\|_{\sigma,\sup}<1$ for any $\sigma:K\to\mathbb{C}$.

Thus we recover a result of Moriwaki (**[20]** Theorem 4.5 (1)$\Longleftrightarrow$(2)).

Corollary 5.6. — *Assume L is big. Then there exists a Hermitian line bundle $\overline{M}$ on $\operatorname{Spec}\mathcal{O}_K$ such that $\overline{\mathcal{L}}\otimes\pi^*\overline{M}$ is arithmetically big.*

6. Continuity of truncated asymptotic polygon

Let us keep the notation of §5 and assume that L is big on X. For any integer $n\ge 1$, denote by ν_n the dilated measure $T_{\frac{1}{n}}\nu_{\pi_*(\overline{\mathcal{L}}^{\otimes n})}$. Recall that in §4 we have actually established the following result:

Proposition 6.1. — *1) the sequence of Borel measures $(\nu_n)_{n\ge1}$ converges vaguely to a Borel probability measure ν;*
2) there exists a countable subset Z of $\mathbb{R}$ such that, for any $\alpha\in\mathbb{R}\setminus Z$, the sequence of polygons $(P(\nu_n^{(\alpha)}))_{n\ge1}$ converges uniformly to $P(\nu^{(\alpha)})$, which implies in particular that $P(\nu^{(\alpha)})$ is Lipschitz.

Let Z be as in the proposition above. For any $\alpha\in\mathbb{R}\setminus Z$, denote by $P_{\overline{\mathcal{L}}}^{(\alpha)}$ the concave function $P(\nu^{(\alpha)})$ on $[0,1]$. The following property of $P_{\overline{\mathcal{L}}}^{(\alpha)}$ results from the definition:

Proposition 6.2. — *For any integer $p\ge1$, on has $P_{\overline{\mathcal{L}}^{\otimes p}}^{(p\alpha)}=pP_{\overline{\mathcal{L}}}^{(\alpha)}$.*

Proof. — By definition $T_{\frac{1}{n}}\nu_{\pi_*(\overline{\mathcal{L}}^{\otimes pn})} = T_p\nu_n$. Using $(T_p\nu_n)^{(p\alpha)} = T_p\nu_n^{(\alpha)}$, we obtain the desired equality. □

Remark 6.3. — We deduce from the previous proposition the equality $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}^{\otimes p}) = p^d\widehat{\mathrm{vol}}(\overline{\mathcal{L}})$, which has been proved by Moriwaki ([**20**] Proposition 4.7).

The main purpose of this section is to establish the following continuity result, which is a generalization of the continuity of the arithmetic volume function proved by Moriwaki (cf. [**20**] Theorem 5.4).

Theorem 6.4. — *Assume $\overline{\mathscr{L}}$ is a Hermitian line bundle on $\mathcal{X}$. Then, for any $\alpha \in \mathbb{R}$, the sequence of functions $\left(\frac{1}{p}P^{(p\alpha)}_{\overline{\mathcal{L}}^{\otimes p}\otimes\overline{\mathscr{L}}}\right)_{p\geq 1}$ converges uniformly to $P^{(\alpha)}_{\overline{\mathcal{L}}}$.*

***Corollary 6.5* ([20] Theorem 5.4).** — *With the assumption of Theorem 6.4, one has*

$$\lim_{p\to\infty}\frac{1}{p^d}\widehat{\mathrm{vol}}(\overline{\mathcal{L}}^{\otimes p}\otimes\overline{\mathscr{L}}) = \widehat{\mathrm{vol}}(\overline{\mathcal{L}}).$$

In order to prove Theorem 6.4, we need the following lemma.

Lemma 6.6. — *Let $\overline{\mathscr{L}}$ be an arbitrary Hermitian line bundle on* Spec $\mathcal{O}_K$. *If $\overline{\mathcal{L}}$ is arithmetically big, then there exists an integer $q \geq 1$ such that $\overline{\mathcal{L}}^{\otimes q}\otimes\overline{\mathscr{L}}$ is arithmetically big and has at least one non-zero* effective *global section, that is, a non-zero section $s \in H^0(\mathcal{X}, \mathcal{L}^{\otimes q}\otimes\mathscr{L})$ such that $\|s\|_{\sigma,\sup} \leq 1$ for any embedding $\sigma : K \to \mathbb{C}$.*

Proof. — As $\overline{\mathcal{L}}$ is arithmetically big, we obtain that L is big on X. Therefore, there exists an integer $m_0 \geq 1$ such that $L^{\otimes m_0}\otimes\mathscr{L}_K$ is big on X and $\pi_*(\mathcal{L}^{\otimes m_0}\otimes\mathscr{L}) \neq 0$. Pick an arbitrary non-zero section $s \in H^0(\mathcal{X}, \mathcal{L}^{\otimes m_0}\otimes\mathscr{L})$ and let $M = \sup_{\sigma:K\to\mathbb{C}}\|s\|_{\sigma,\sup}$. After Theorem 5.4 (see also Remark 5.5), there exists $m_1 \in \mathbb{N}$ such that $\mathcal{L}^{\otimes m_1}$ has a section s' such that $\|s'\|_{\sigma,\sup} \leq (2M)^{-1}$ for any $\sigma : K \to \mathbb{C}$. Let $q = m_0 + m_1$. Then $s \otimes s'$ is a non-zero strictly effective section of $\overline{\mathcal{L}}^{\otimes q}\otimes\overline{\mathscr{L}}$. Furthermore, $\overline{\mathcal{L}}^{\otimes q}\otimes\overline{\mathscr{L}}$ is arithmetically big since it is generically big and has a strictly effective section. □

Proof of Theorem 6.4. — After Corollary 5.6, we may assume that $\overline{\mathcal{L}}$ is arithmetically big. Let $q \geq 1$ be an integer such that $\overline{\mathcal{L}}^{\otimes q}\otimes\overline{\mathscr{L}}$ is arithmetically big and has a non-zero effective section s_1 (cf. Lemma 6.6). For any integers p and n such that $p > q$, $n \geq 1$, let $\varphi_{p,n} : \pi_*(\mathcal{L}^{\otimes(p-q)n}) \to \pi_*(\mathcal{L}^{\otimes pn}\otimes\mathscr{L}^{\otimes n})$ be the homomorphism defined by the multiplication by $s_1^{\otimes n}$. Since s_1 is effective, $h(\varphi_{p,n}) \leq 0$. Let

$$\theta_{p,n} = \mathrm{rk}(\pi_*(\mathcal{L}^{\otimes(p-q)n}))/\,\mathrm{rk}(\pi_*(\mathcal{L}^{\otimes pn}\otimes\mathscr{L}^{\otimes n})).$$

Note that

$$\lim_{n\to\infty}\theta_{p,n} = \mathrm{vol}(L^{\otimes(p-q)})/\mathrm{vol}(L^{\otimes p}\otimes\mathscr{L}_K).$$

Denote by θ_p this limit. Let $\nu_{p,n}$ be the measure associated to $\pi_*(\overline{\mathcal{L}}^{\otimes pn}\otimes\overline{\mathscr{L}}^{\otimes n})$. Let $a_{p,n} = \widehat{\mu}_{\min}(\pi_*(\overline{\mathcal{L}}^{\otimes pn}\otimes\overline{\mathscr{L}}^{\otimes n}))$. After Proposition 2.2, one has $\nu_{p,n} \succ$

$\theta_{p,n}T_{(p-q)n}\nu_{(p-q)n} + (1-\theta_{p,n})\delta_{a_{p,n}}$, or equivalently

$$T_{\frac{1}{np}}\nu_{p,n} \succ \theta_{p,n}T_{(p-q)/p}\nu_{(p-q)n} + (1-\theta_{p,n})\delta_{a_{p,n}/np}. \tag{27}$$

As $L^{\otimes p}\otimes \mathscr{L}_K$ is big, the sequence of measures $(T_{\frac{1}{n}}\nu_{p,n})_{n\geq 1}$ converges vaguely to a Borel probability measure η_p. By truncation and then by passing $n\to\infty$, we obtain from (27) that, for any $\alpha\in\mathbb{R}$,

$$(T_{\frac{1}{p}}\eta_p)^{(\alpha)} \succ \theta_p(T_{(p-q)/p}\nu)^{(\alpha)} + (1-\theta_p)\delta_\alpha, \tag{28}$$

where we have used the trivial estimation $\delta_a^{(\alpha)}\succ\delta_\alpha$.

Now we apply Lemma 6.6 on the dual Hermitian line bundle $\overline{\mathscr{L}}^\vee$ and obtain that there exists an integer $r\geq 1$ and an effective section s_2 of $\overline{\mathcal{L}}^{\otimes r}\otimes\overline{\mathscr{L}}^\vee$. Consider the homomorphism $\psi_{p,n}:\pi_*(\mathcal{L}^{\otimes pn}\otimes\mathscr{L}^{\otimes n})\to\pi_*(\mathcal{L}^{\otimes(p+r)n})$ induced by multiplication by $s_2^{\otimes n}$. Its height is negative. Let

$$\vartheta_{p,n} = \operatorname{rk}(\pi_*(\mathcal{L}^{\otimes pn}\otimes\mathscr{L}^{\otimes n}))/\operatorname{rk}(\pi_*(\mathcal{L}^{\otimes(p+r)n})).$$

When n tends to infinity, $\vartheta_{p,n}$ converges to

$$\vartheta_p := \operatorname{vol}(L^{\otimes p}\otimes\mathscr{L}_K)/\operatorname{vol}(L^{\otimes(p+r)}).$$

By a similar argument as above, we obtain that, for any $\alpha\in\mathbb{R}$,

$$(T_{(p+r)/p}\nu)^{(\alpha)} \succ \vartheta_p(T_{\frac{1}{p}}\eta_p)^{(\alpha)} + (1-\vartheta_p)\delta_\alpha. \tag{29}$$

We obtain from (28) and (29) the following estimations of polygons

$$\vartheta_p^{-1}P((T_{(p+r)/p}\nu)^{(\alpha)})(\vartheta_p t) \geq P((T_{\frac{1}{p}}\eta_p)^{(\alpha)})(t) \tag{30}$$

$$P((T_{\frac{1}{p}}\eta_p)^{(\alpha)})(t) \geq \begin{cases}\theta_p P((T_{(p-q)/p}\nu)^{(\alpha)})(t/\theta_p), & 0\leq t\leq\theta_p,\\ \theta_p P((T_{(p-q)/p}\nu)^{(\alpha)})(1)+\alpha(t-\theta_p), & \theta_p\leq t\leq 1.\end{cases} \tag{31}$$

Finally, since $\lim\limits_{p\to\infty}\theta_p = \lim\limits_{p\to\infty}\vartheta_p = 1$ (which is a consequence of the continuity of the geometric volume function), combined with the fact that both $T_{(p-q)/p}\nu$ and $T_{(p+r)/p}\nu$ converge vaguely to ν when $p\to\infty$, we obtain, for any $\alpha\in\mathbb{R}$, the uniform convergence of $P((T_{\frac{1}{p}}\eta_p)^{(\alpha)})$ to $P(\nu^{(\alpha)})$. □

7. Computation of asymptotic polygon by volume function

In this section we shall show how to compute the asymptotic polygon of a Hermitian line bundle by using arithmetic volume function. Our main method is the Legendre transformation of concave functions. Let us begin with a lemma concerning Borel measures.

Lemma 7.1. — *Let ν be a Borel measure on $\mathbb{R}$ whose support is bounded from below. Then*

$$\max_{t\in[0,1[} P(\nu)(t) = \int_{\mathbb{R}} x_+\nu(\mathrm{d}x), \tag{32}$$

where x_+ stands for $\max\{x,0\}$.

Proof. — The function F_ν^* defined in §2.3 is essentially the inverse of the distribution function of ν. In fact, one has $F_\nu^*(F_\nu(x)) = x \quad \nu$-a.e.. Therefore, if η is a Borel measure of compact support, then

$$P(\eta)(1) := \int_0^1 F_\eta(t)\,\mathrm{d}t = \int_{\mathbb{R}} x\eta(\mathrm{d}x).$$

Applying this equality on $\eta = \nu^{(0)}$, we obtain

$$\max_{t\in[0,1[} P(\nu)(t) = P(\nu^{(0)})(1) = \int_{\mathbb{R}} x\nu^{(0)}(\mathrm{d}x) = \int_{\mathbb{R}} x_+\nu^{(0)}(\mathrm{d}x) = \int_{\mathbb{R}} x_+\nu(\mathrm{d}x).$$

□

Now let $\mathscr{X}$ be an arithmetic variety of total dimension d. For any Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathscr{X}$ whose generic fibre is big, we denote by $\nu_{\overline{\mathcal{L}}}$ the vague limite of the sequence of measures $(T_{\frac{1}{n}}\nu_{\pi_*(\overline{\mathcal{L}}^{\otimes n})})_{n\geq 1}$. The existence of $\nu_{\overline{\mathcal{L}}}$ has been established in Theorem 4.2.

Proposition 7.2. — *Let $L = \mathcal{L}_K$. For any real number a, one has*

$$\int_{\mathbb{R}} (x-a)_+\nu_{\overline{\mathcal{L}}}(\mathrm{d}x) = \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{L}}\otimes\pi^*\overline{\mathscr{L}}_{-a})}{d\mathrm{vol}(L)},$$

where $\overline{\mathscr{L}}_{-a}$ is the Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$ *defined in* (3).

Proof. — If $\overline{M}$ is a Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$, one has the equality

$$\nu_{\overline{\mathcal{L}}\otimes\pi^*\overline{M}} = \tau_{\widehat{\deg}(\overline{M})}\nu_{\overline{\mathcal{L}}}.$$

Applying this equality on $\overline{M} = \overline{\mathscr{L}}_{-a}$, one obtains

$$\frac{\widehat{\mathrm{vol}}(\overline{\mathcal{L}}\otimes\pi^*\overline{\mathscr{L}}_{-a})}{d\mathrm{vol}(L)} = \int_{\mathbb{R}} x_+\tau_{-a}\nu_{\overline{\mathcal{L}}}(\mathrm{d}x) = \int_{\mathbb{R}} (x-a)_+\nu_{\overline{\mathcal{L}}}(\mathrm{d}x).$$

□

Remark 7.3. — Proposition 7.2 calculates actually the polygon $P(\nu_{\overline{\mathcal{L}}})$. In fact, one has

$$-\int_a^{+\infty} \nu_{\overline{\mathcal{L}}}(]y,+\infty[)\mathrm{d}y = -\int_{\mathbb{R}}(s-a)_+\nu_{\overline{\mathcal{L}}}(\mathrm{d}s).$$

Applying the Legendre transformation, we obtain the polygon $P(\nu_{\overline{\mathcal{L}}})$.

As an application, we prove that the asymptotic maximal slope of $\overline{\mathcal{L}}$ is in fact the derivative at the origin of the concave curve $P(\nu_{\overline{\mathcal{L}}})$, which is however not a formal consequence of the vague convergence of measures $T_{\frac{1}{n}}\nu_{\pi_*(\overline{\mathcal{L}}^{\otimes n})}$.

Proposition 7.4. — *Denote by $\widehat{\mu}_{\max}^{\pi}(\overline{\mathcal{L}})$ the limite of $(\widehat{\mu}_{\max}(\overline{\mathcal{L}}^{\otimes n})/n)_{n\geq 1}$. Then one has*

$$P(\nu_{\overline{\mathcal{L}}})'(0) = \widehat{\mu}_{\max}^{\pi}(\overline{\mathcal{L}}).$$

Proof. — For any integer $n \geq 1$, let $\nu_n = T_{\frac{1}{n}}\nu_{\pi_*(\overline{\mathcal{L}}^{\otimes n})}$. Since $P(\nu_n)$ is a concave curve on $[0,1]$, for any $t \in]0,1]$, one has

$$\frac{\widehat{\mu}_{\max}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))}{n} = P(\nu_n)'(0) \geq \frac{P(\nu_n)(t)}{t}.$$

As the sequence of polygons $P(\nu_n)$ converges uniformly to $P(\nu)$, we obtain

$$\widehat{\mu}_{\max}^{\pi}(\overline{\mathcal{L}}) \geq P(\nu)(t)/t, \qquad \forall t \in]0,1].$$

Therefore $\widehat{\mu}_{\max}^{\pi}(\overline{\mathcal{L}}) \geq P(\nu)'(0)$.

Let $\varepsilon > 0$ be an arbitrary positive real number. Let $a = \widehat{\mu}_{\max}^{\pi}(\overline{L}) - \varepsilon$. Then $\widehat{\mu}_{\max}^{\pi}(\overline{\mathcal{L}} \otimes \pi^*\overline{\mathscr{L}}_{-a}) = \varepsilon > 0$. Therefore, Lemma 7.1, Proposition 7.2 and the first part of Theorem 5.4 implies that

$$\max_{t\in[0,1]}\Big(P(\nu)(t) - at\Big) = \int_{\mathbb{R}} (x-a)_+ \nu_{\overline{\mathcal{L}}}(\mathrm{d}x) > 0.$$

Hence

$$P(\nu)'(0) = \max_{t\in]0,1]} \frac{P(\nu)(t)}{t} \geq \max_{t\in[0,1]}\Big(P(\nu)(t) - at\Big) + a > a.$$

Since ε is arbitrary, we obtain $P(\nu)'(0) \geq \widehat{\mu}_{\max}^{\pi}(\overline{\mathcal{L}})$. □

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Huayi Chen, CMLS, École Polytechnique, 91120, Palaiseau, France. Université Paris 8 • *E-mail :* `huayi.chen@polytechnique.org`
Url : `http://www.math.polytechnique.fr/~chen`