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Distribution of logarithmic spectra of the equilibrium energy

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Abstract. Let L be a big invertible sheaf on a projective variety defined on a complete valued field (such as the field $\mathbb{C}$ of complex numbers or a complete non-archimedean field), equipped with two continuous metrics. By using the ideas in Arakelov geometry, we prove that the distribution of the eigenvalues of the transition matrix between the L^2 norms on $H^0(X, nL)$ with respect to the two metriques converges (in law) as n goes to infinity to a Borel probability measure on $\mathbb{R}$. This result can be thought of as a generalization of the existence of the energy at the equilibrium as a limit, or an extension of Berndtsson's results to the more general context of graded linear series and a more general class of line bundles.

Contents

1. Introduction

Let K be a number field and X be an integral projective scheme defined over K. Assume given an adelic line bundle $\overline{L}$ on X. Namely L is an invertible sheaf on X equipped with a family $(h_v)_{v\in M_K}$ of metrics indexed by the set M_K of all places of K, where each h_v is a continuous metric on the analytification of L with respect to the place v. For each $v \in M_K$ and any integer $n \geqslant 0$, the metric h_v induces naturally a norm on $V_n := H^0(X, L^{\otimes n})$, which is the sup norm when v is non-archimedean, and is the L^2 norm with respect to a Borel probability measure

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which is equivalent to the Lebesgue measure on each local coordinate chart when v is archimedean. Thus the K-vector space V_n is equipped with an adelic structure and becomes an adelic vector bundle on Spec K. Recall that the arithmetic Hilbert–Samuel theorem [1] describes the asymptotic behaviour of the normalized Arakelov degree of $\overline{V}_n$ when $n \to +\infty$. Note that the Arakelov degree of a hermitian adelic vector bundle on Spec K equals the sum of successive slopes of the adelic vector bundle. In [10], a limit theorem has been established, asserting that the uniform distribution of the suitably normalized successive slopes of $\overline{V}_n$ converges in law to some Borel probability measure on $\mathbb{R}$. The method was based on a convexity property of successive slopes of $\overline{V}_n$ and the functoriality of Harder–Narasimhan filtrations.

The current paper treats a local analogue of the above result. We focus on the situation where the scheme is defined on a complete valued field. The metric structure is given by two continuous metrics on the same invertible sheaf. The purpose is to compare the sup norms or the L^2 norms induced by these two metrics on the linear series of the invertible sheaf (or its tensor powers). This problem has been widely studied in the framework of complex hermitian geometry, notably in [3,4,20].

Let X be a complex projective variety or alternatively, a projective variety defined over a complete non-archimedean field, and let L be an invertible sheaf on X. We assume that L is *big*, or in other words, that the *volume* of L, defined by

$$\mathrm{vol}(L) := \lim_{n\to+\infty} \frac{\mathrm{rk}(H^0(X, nL))}{n^d/d!},$$

is strictly positive, where the rank is computed with respect to the base field of X, and d is the Krull dimension of X. In [3], Berman and Boucksom studied in the complex setting the Monge–Ampère energy functional of two continuous metrics on L, φ and ψ. The equilibrium Monge–Ampère energy of the pair (φ, ψ) is an invariant defined by

$$\mathcal{E}_{\mathrm{eq}}(\varphi, \psi) := \frac{1}{d+1} \sum_{j=0}^{d} \int_{X(\mathbb{C})} (P\varphi - P\psi)\, c_1(L, P\psi)^j c_1(L, P\varphi)^{d-j}.$$

In the above formula $P\phi$ denotes, for any continuous metric ϕ on L, the supremum of all (possibly singular) plurisubharmonic continuous metrics bounded above by ϕ. This invariant describes the asymptotic behaviour of the ratio of the volumes of the unit balls in the linear systems $H^0(X, nL)$ $(n \geqslant 1)$ with respect to the L^2 norms induced by the metrics φ and ψ, respectively. More precisely, given any probability measure μ on $X(\mathbb{C})$, equivalent to Lebesgue measure in every local chart, we have that (cf. [3, Théorème A])

$$\lim_{n\to+\infty} \frac{d!}{n^{d+1}} \ln \frac{\mathrm{vol}(\mathcal{B}^2(nL, \varphi, \mu))}{\mathrm{vol}(\mathcal{B}^2(nL, \psi, \mu))} = \mathcal{E}_{\mathrm{eq}}(\varphi, \psi), \tag{1}$$

where for any continuous metric ϕ on L, the expression $\mathcal{B}^2(nL, \phi, \mu)$ denotes the unit ball in $H^0(X, nL)$ with respect to the following L^2 norm:

$$\forall\, s \in H^0(X, nL), \quad \|s\|^2_{L^2_{\phi,\mu}} := \int_{X(\mathbb{C})} \|s\|^2_{n\phi}(x)\, \mu(\mathrm{d}x).$$

All volumes are calculated with respect to some arbitrary Haar measure on $H^0(X, nL)$.

The logarithm which appears in the left hand side of Eq. (1) is equal (up to multiplication by a constant) to the mean of the logarithms of the eigenvalues of the transition matrix between φ_n and ψ_n. Here φ_n and ψ_n are the L^2 norms induced by φ and ψ respectively on the space $H^0(X, nL)$.

In [4], Berndtsson studies the distribution of eigenvalues of this transition matrix in the case where φ and ψ are Kähler metrics on an ample line bundle L. He establishes the following fact by a detailed analysis of geodesics in the space of Kähler metrics on L : if $d_n = \dim V_n$ and the numbers λ_j are the logarithms of the eigenvalues of the transition matrices between φ_n and ψ_n then the probability measure

$$\nu_n = d_n^{-1} \sum_j \delta_{\lambda_j}$$

converges as n tends to infinity to a measure on $\mathbb{R}$ which is defined using the Monge-Ampère geodesic linking φ and ψ in $\mathcal{H}_L$, the space of Hermitian metrics in L.

The log-ratio of the volumes appearing in (1) is an analogue of the degree function in Arakelov geometry. Let E be a vector space of finite rank over a number field K equipped with a family of norms $\|.\|_v$, where v runs over the set M_K of places of K, and $\|.\|_v$ is a norm on $E \otimes_K K_v$ which is ultrametric when v is a finite place and Euclidean or Hermitian when v is real or complex. Moreover, we suppose that E contains a lattice $\mathcal{E}$ (ie. a maximal rank sub-$\mathcal{O}_K$-module, where $\mathcal{O}_K$ is the ring of algebraic integers in K) such that for all but a finite number of places v the norm $\|.\|_v$ comes from the $\mathcal{O}_K$-module structure on $\mathcal{E}$.

The data $\overline{E} = (E, (\|.\|_v)_{v \in M_K})$ is called a *Hermitian adelic bundle* on K and its *Arakelov degree* is defined to be the weighted sum

$$\widehat{\deg}(\overline{E}) := - \sum_{v \in M_K} n_v \ln \|s_1 \wedge \cdots \wedge s_r\|_v, \tag{2}$$

where $(s_1, \ldots, s_r)$ is a basis of E over K and the weight n_v is the rank of K_v as a vector space over $\mathbb{Q}_v$. The Arakelov degree is well-defined due to the product formula

$$\forall a \in K^\times, \quad \sum_{v \in M_K} n_v \ln |a|_v = 0.$$

We refer the reader to [5, Appendice A] and [13] for a detailed exposition of this theory. In the context of complex geometry, we consider a finite dimensional vector space V equipped with two Hermitian norms φ and ψ. We can write the log-ratio of the volumes of the unit balls (with respect to φ and ψ respectively) as

$$\ln \frac{\mathrm{vol}(\mathcal{B}(V, \varphi))}{\mathrm{vol}(\mathcal{B}(V, \psi))} = \widehat{\deg}(V, \varphi, \psi) := - \ln \|s_1 \wedge \cdots \wedge s_r\|_\varphi + \ln \|s_1 \wedge \cdots \wedge s_r\|_\psi, \tag{3}$$

where $(s_1, \ldots, s_r)$ is a basis for V. This expression is independent of the choice of basis by the elementary product formula

$$\forall\, a \in \mathbb{C}^{\times}, \quad \ln|a| - \ln|a| = 0.$$

With this notation, Eq. (1) can be rewritten in the form

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(H^0(X, nL), L^2_{\varphi,\mu}, L^2_{\psi,\mu})}{n^{d+1}/d!} = \mathcal{E}_{\mathrm{eq}}(\varphi, \psi).$$

which is an analogue of the arithmetic Hilbert–Samuel theorem for Hermitian invertible bundles on a projective arithmetic variety.

Given the results in [4] and the convergences proved in [10], it is natural to wonder whether the eigenvalue distribution of the metric $L^2_{\varphi,\mu}$ with respect to $L^2_{\psi,\mu}$ is well-behaved asymptotically in more general situations. Unlike the arithmetic case, it does not seem possible to reformulate these spectra functorially, which is a key step in the proof of the convergence results in [10]. This phenomenon arises essentially because of the presence of a negative weight in formula (3), absent in the arithmetic case. The interested reader will find counter-examples illustrating this phenomenon in [11, remark 5.10 and §4, example 2].

In this article, we use a truncation method for studying the asymptotic behaviour of eigenvalues of transition matrices between two metrics. Note that similar idea appears also in the study of linear forms of logarithms (see [14, §4.1]) We consider truncations of ψ by dilatations of φ, which enables us to prove the following result (cf. theorem 5.2 *infra*):

Main theorem. *Let k be the field of complex numbers or a complete non-archimedean field, and let X be a projective k-variety of dimension $d \geqslant 1$. Let L be a big invertible $\mathcal{O}_X$-module with two continuous metrics φ and ψ. For any integer $n \geqslant 1$, let φ_n and ψ_n be the sup norms on $H^0(X, nL)$ with respect to metrics φ and ψ respectively. Moreover, let φ'_n and ψ'_n be hermitian norms on $H^0(X, nL)$ such that*[1] $\max(d(\varphi_n, \varphi'_n), d(\psi_n, \psi'_n)) = o(n)$. *Let Z_n be the map from $\{1, \ldots, h^0(X, nL)\}$ to $\mathbb{R}$ sending i to the logarithm of the i^{th} eigenvalue (with multiplicity) of ψ'_n with respect to φ'_n, considered as a random variable on the set $\{1, \ldots, h^0(X, nL)\}$ with the uniform distribution. Then the sequence of random variables $(Z_n/n)_{n\geqslant 1}$ converges in law to a probability distribution on $\mathbb{R}$ depending only on the pair (φ, ψ).*

We recall that by definition the sequence of random variables Z_n/n converges in law if for any continuous bounded function h defined on $\mathbb{R}$ the sequence $(\mathbb{E}[h(Z_n/n)])_{n\geqslant 1}$ converges in $\mathbb{R}$.

The Theorem 5.2 proved in Sect. 5 is a little bit stronger than this statement. It is valid for any sub-graded linear system of subspaces $V_n \subset H^0(X, nL)$ satisfying conditions (a)–(c) of Sect. 4.3. Moreover, it also applies to the functions $\widehat{\mu}_i$ defined in Sect. 2.3, and which are associated to pairs of (possibly non-hermitian) norms.

[1] If η and η' are two norms on a finite-dimensional complex vector space V, then $d(\eta, \eta')$ is defined to be $\sup\limits_{0\neq s\in V} |\ln\|s\|_\eta - \ln\|s\|_{\eta'}|$. This is a distance on the set of norms on V.

When these norms are in fact hermitian, these functions are equal to the logarithms of the eigenvalues of the transition matrix (cf. Sect. 2.2). This asymptotic distribution is a fine geometric invariant which measures the degree of non-proportionality of the metrics φ and ψ. It should be useful in the variational study of metrics on an invertible sheaf.

The proof of the main theorem uses various techniques drawn from algebraic and arithmetic geometry. In Sects. 2 and 3 we introduce a Harder–Narasimhan type theory for finite dimensional vector spaces with two norms. This can be thought of as a geometric reformulation of the eigenvalues of the transition matrix between two Hermitian norms. This construction has the advantage of being valid for non-Hermitian norms, which allows us to work directly with the sup norm. Moreover, it is analogous to the Harder–Narasimhan filtration and polygon of a vector bundle on a smooth projective curve. In particular, the truncation results (Propositions 2.8, 3.7) are crucial for the proof of the main theorem. Another important ingredient is the existence of the equilibrium energy as a limit, presented in Sect. 4. For a complete complex graded system, this is a result of Berman and Boucksom [3] (where the limit is described.) Here, we use the Okounkov filtered semi-groups point of view developped in [7], analogous to that of Witt Nyström [20]. The combination of this method with the Harder–Narasimhan formalism is extremely flexible and enables us to prove the existence of the equilibrium energy as a limit in the very general setting of a graded linear system on both complex and non-archimedean varieties (cf. Theorem 4.5 and its Corollary 4.6). Finally, in Sect. 5 we prove a general version of the main theorem (cf. Theorem 5.2 and Remark 5.3).

2. Slopes of a vector space equipped with two norms

In this section we develop the formalism of slopes and Harder–Narasimham filtrations for finite dimensional complex vector spaces equipped with a pair of norms.

2.1. Slopes and the Harder–Narasimhan filtration

Let $\mathcal{C}^H$ be the class of triplets $\overline{V} = (V, \varphi, \psi)$, where V is a finite dimensional complex vector space and φ and ψ are two Hermitian norms on V. For any $\overline{V} = (V, \varphi, \psi) \in \mathcal{C}^H$, we let $\widehat{\deg}(\overline{V})$ be the real number defined by

$$- \ln \|s_1 \wedge \cdots \wedge s_r\|_\varphi + \ln \|s_1 \wedge \cdots \wedge s_r\|_\psi, \tag{4}$$

where $(s_1, \ldots, s_r)$ is a basis of V. Note that when V is the trivial vector space, one has $\widehat{\deg}(\overline{V}) = 0$. If the V is non trivial then we let $\widehat{\mu}(\overline{V})$ be the quotient $\widehat{\deg}(\overline{V})/\operatorname{rk}(V)$, which we call the *slope* of $\overline{V}$. Unless otherwise specified, for any subspace W of V we will let $\overline{W}$ be the vector space W equipped with the induced norms and $\overline{V}/\overline{W}$ be the quotient space V/W equipped with the quotient norms. The following relationship holds for any subspace W of V:

$$\widehat{\deg}(\overline{V}) = \widehat{\deg}(\overline{W}) + \widehat{\deg}(\overline{V}/\overline{W}). \tag{5}$$

Proposition 2.1. *Let $\overline{V} = (V, \varphi, \psi)$ be a non-trivial element of $\mathcal{C}^H$. There is a unique subspace V_{des} in V satisfying the following properties:*

(1) for any subspace $W \subset V$ we have that $\widehat{\mu}(\overline{W}) \leqslant \widehat{\mu}(\overline{V}_{\text{des}})$,
(2) if W is a subspace of V such that $\widehat{\mu}(\overline{W}) = \widehat{\mu}(\overline{V}_{\text{des}})$ then $W \subset V_{\text{des}}$.

Proof. Let λ be the norm of the identity map from $(V, \|.\|_\varphi)$ to $(V, \|.\|_\psi)$. We will prove that the set

$$V_{\text{des}} = \{x \in V \;:\; \|x\|_\psi = \lambda \|x\|_\varphi\}$$

satisfies the conditions of the proposition. We start by checking that V_{des} is a non-trivial subspace of V. By definition, V_{des} contains at least one non-zero vector and is stable under multiplication by a complex scalar. We need to check that V_{des} is stable under addition. Let x and y be two elements of V_{des}. As the norms φ and ψ are Hermitian, the parallelogram law states that

$$\|x+y\|_\varphi^2 + \|x-y\|_\varphi^2 = 2\left(\|x\|_\varphi^2 + \|y\|_\varphi^2\right),$$
$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right).$$

Since λ is the norm of the identity map from $(V, \|.\|_\varphi)$ to $(V, \|.\|_\psi)$ we have that $\|x+y\|_\psi \leqslant \lambda\|x+y\|_\varphi$ and $\|x-y\|_\psi \leqslant \lambda\|x-y\|_\varphi$. As x and y are vectors in V_{des} we have that $\|x\|_\psi = \lambda\|x\|_\varphi$ and $\|y\|_\psi = \lambda\|y\|_\varphi$. It follows that

$$\begin{aligned} 2\lambda^2\left(\|x\|_\varphi^2 + \|y\|_\varphi^2\right) &= 2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right) = \|x+y\|_\psi^2 + \|x-y\|_\psi^2 \\ &\leqslant \lambda^2\left(\|x+y\|_\varphi^2 + \|x-y\|_\varphi^2\right) = 2\lambda^2\left(\|x\|_\varphi^2 + \|y\|_\varphi^2\right), \end{aligned}$$

so $\|x+y\|_\psi = \lambda\|x+y\|_\varphi$ and it follows that $x + y \in V_{\text{des}}$.

In particular, as the norms $\|.\|_\psi$ and $\|.\|_\varphi$ are proportional with ratio λ on V_{des} we have that $\widehat{\mu}(\overline{V}_{\text{des}}) = \ln(\lambda)$. If W is a subspace of V then the identity map from $(\Lambda^r W, \|.\|_\psi)$ to $(\Lambda^r W, \|.\|_\varphi)$ has norm $\leqslant \lambda^r$ (by Hadamard's inequality). This inequality is an equality if and only if the norms $\|.\|_\psi$ and $\|.\|_\varphi$ are proportional with ratio λ—or in other words, the space W is contained in V_{des}. $\square$

Let $\overline{V}$ be a non-trivial element of $\mathcal{C}^H$. Let $\widehat{\mu}_{\max}(\overline{V})$ be the slope of $\overline{V}_{\text{des}}$, which we call the *maximal slope* of $\overline{V}$. We say that $\overline{V}$ is *semi-stable* if $\widehat{\mu}_{\max}(\overline{V}) = \widehat{\mu}(\overline{V})$, or equivalently $V_{\text{des}} = V$. The element $\overline{V}$ is semi-stable if and only if the two norms on $\overline{V}$ are proportional.

For any $\overline{V}$, non-trivial element of $\mathcal{C}^H$, we construct recursively a sequence of subspaces of V of the form

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \tag{6}$$

such that $V_i/V_{i-1} = (V/V_{i-1})_{\text{des}}$ for any $i \in \{1, \ldots, n\}$ using the quotient norms on V/V_{i-1}. Each of the subquotients $\overline{V}_i/\overline{V}_{i-1}$ is a semi-stable element of $\mathcal{C}^H$. Moreover, if μ_i is the slope of $\overline{V}_i/\overline{V}_{i-1}$, then we have that:

$$\mu_1 > \mu_2 > \cdots > \mu_n. \tag{7}$$

These numbers are called the *intermediate slopes* of $\overline{V}$. The flag (6) is called the *Harder–Narasimhan filtration* of $\overline{V}$.

Definition 2.2. Let $\overline{V}$ be a non-zero element of $\mathcal{C}^H$ with its Harder–Narasimhan filtration and intermediate slopes as defined in (6) and (7). We let $Z_{\overline{V}}$ be the random variable with values in $\{\mu_1, \ldots, \mu_n\}$ such that

$$\mathbb{P}(Z_{\overline{V}} = \mu_i) = \frac{\mathrm{rk}(V_i/V_{i-1})}{\mathrm{rk}(V)}$$

for any $i \in \{1, \ldots, n\}$.

Remark 2.3. The above constructions are analogues of Harder–Narasimhan theory for vector bundles on a regular projective curve or hermitian adelic bundles on a number field. As in [10, §2.2.2], we can include the Harder–Narasimhan filtration and the intermediate slopes in a decreasing $\mathbb{R}$-filtration. However, contrary to the geometric and arithmetic cases, this $\mathbb{R}$-filtration is not functorial, as explained in [11, remarque 5.10]. Moreover, the Harder–Narasimhan filtration is not necessarily the only filtration whose sub-quotients are semi-stable with strictly decreasing slopes, as can be seen using the counter-example in [11, §4 exemple 2].

Let $\overline{V}$ be an element of $\mathcal{C}^H$ of rank $r > 0$. We let $\widetilde{P}_{\overline{V}}$ be the function on $[0, r]$ whose graph is the upper boundary of the convex closure of the set of points of the form $(\mathrm{rk}(W), \widehat{\deg}(\overline{W}))$. This is a concave function which is affine on each interval $[i-1, i]$ ($i \in \{1, \ldots, r\}$). We call it the Harder–Narasimhan *polygon* of $\overline{V}$. By definition, $\widetilde{P}_{\overline{V}}(0) = 0$ and $\widetilde{P}_{\overline{V}}(r) = \widehat{\deg}(V)$. For any $i \in \{1, \ldots, r\}$, we let $\widehat{\mu}_i(\overline{V})$ be the slope of the function $\widetilde{P}_{\overline{V}}$ on the interval $[i-1, i]$, which we call the i^{th} *slope* of $\overline{V}$. We also introduce a normalised version of the Harder–Narasimhan polygon, defined by

$$P_{\overline{V}}(t) := \frac{1}{\mathrm{rk}(V)} \widetilde{P}_{\overline{V}}(t\,\mathrm{rk}(V)), \quad t \in [0, 1].$$

It follows from the relation $\widetilde{P}_{\overline{V}}(r) = \widehat{\deg}(V)$ that

$$\widehat{\deg}(\overline{V}) = \widehat{\mu}_1(\overline{V}) + \cdots + \widehat{\mu}_r(\overline{V}). \tag{8}$$

Moreover, the distribution of the random variable $Z_{\overline{V}}$ (cf. Definition 2.2) is given by

$$\frac{1}{r}\sum_{i=1}^{r} \delta_{\widehat{\mu}_i(\overline{V})},$$

where δ_a is a Dirac measure supported at a.

The normalised polygon $P_{\overline{V}}$, the random variable $Z_{\overline{V}}$ and the slopes of $\overline{V}$ are linked by the following simple formula.

$$P_{\overline{V}}(1) = \mathbb{E}[Z_{\overline{V}}] = \widehat{\mu}(\overline{V}). \tag{9}$$

2.2. The link with eigenvalues.

The Harder–Narasimhan polygon and its intermediate slopes can be thought of as an intrinsic interpretation of the eigenvalues and eigenspaces of the transition matrix between two Hermitian norms. Indeed, given an element $\overline{V} = (V, \varphi, \psi) \in \mathcal{C}^H$ and an orthonormal basis $\boldsymbol{e} = (e_i)_{i=1}^r$ for V with respect to $\|.\|_\varphi$ we can construct a Hermitian matrix

$$A_{\overline{V},\boldsymbol{e}} = \begin{pmatrix} \langle e_1, e_1\rangle_\psi & \langle e_1, e_2\rangle_\psi & \cdots & \langle e_1, e_r\rangle_\psi \\ \langle e_2, e_1\rangle_\psi & \langle e_2, e_2\rangle_\psi & \cdots & \langle e_1, e_r\rangle_\psi \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_r, e_1\rangle_\psi & \langle e_r, e_2\rangle_\psi & \cdots & \langle e_r, e_r\rangle_\psi \end{pmatrix}$$

If the intermediate slopes of $\overline{V}$ are $\mu_1 > \mu_2 > \cdots > \mu_n$ then the eigenvalues of $A_{\overline{V},\boldsymbol{e}}$ are $\mathrm{e}^{2\mu_1}, \ldots, \mathrm{e}^{2\mu_n}$. In fact, although the matrix $A_{\overline{V},\boldsymbol{e}}$ depends on the choice of the orthonormal basis $\boldsymbol{e}$, its eigenvalues (counting the multiplicities) do not depend on this choice. Therefore, without loss of generality we may assume that $\boldsymbol{e}$ is an orthogonal basis for the hermitian product $\langle , \rangle_\psi$. In this case the matrix $A_{\overline{V},\boldsymbol{e}}$ is diagonal and its eigenvalues are $(\|e_i\|_\psi^2)_{i=1}^r$. Moreover, from the proof of Proposition 2.1, we learn that V_{des} coincides with the set of vectors $x \in V$ such that $\|x\|_\psi$ equals the operator norm of $\mathrm{Id} : (V, \|.\|_\varphi) \to (V, \|.\|_\psi)$ times $\|s\|_\varphi$. Namely V_{des} is the vector subspace generated by those $e_i \in \boldsymbol{e}$ such that $\|e_i\|_\psi$ is maximal. In particular, the largest eigenvalue of $A_{\overline{V},\boldsymbol{e}}$ identifies with $\mathrm{e}^{2\mu_1}$. By induction we obtain the correspondance between successive slopes of $\overline{V}$ and the eigenvalues of $A_{\overline{V},\boldsymbol{e}}$ as stated above. Moreover, if for any $i \in \{1, \ldots, n\}$ we let $V^{(i)}$ be the eigenspace associated to the eigenvalue $\mathrm{e}^{2\mu_i}$ of the endomorphism of V given by the matrix $A_{\overline{V},\boldsymbol{e}}$ in the basis $\boldsymbol{e}$ then the flag

$$0 \subsetneq V^{(1)} \subsetneq V^{(1)} + V^{(2)} \subsetneq \cdots \subsetneq V^{(1)} + \cdots + V^{(n)} = V$$

is the Harder–Narasimhan filtration of $\overline{V}$.

The fact that φ and ψ are proportional on each of the subspaces $V^{(i)}$ implies that there is a complete flag (which will be in general finer than the Harder–Narasimhan filtration of $\overline{V}$)

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$$

such that $\widehat{\deg}(\overline{V}_i/\overline{V}_{i-1}) = \widehat{\mu}_i(\overline{V})$. In particular, $P_{\overline{V}}(i) = \widehat{\deg}(\overline{V}_i)$. This can be thought of as a version of the Courant–Fischer theorem for symmetric or Hermitian matrices.

2.3. Generalisation to arbitrary norms

The above constructions can be naturally generalised to spaces equipped with two arbitrary norms. Let $\mathcal{C}$ be the class of finite-dimensional complex vector spaces equipped with two (not necessarily Hermitian) norms. For any non-trivial element $\overline{V} = (V, \varphi, \psi)$ in $\mathcal{C}$ we let $\widehat{\deg}(\overline{V})$ be the number

$$\ln \frac{\mathrm{vol}(\mathcal{B}(V, \varphi))}{\mathrm{vol}(\mathcal{B}(V, \psi))},$$

where $\mathcal{B}(V, \varphi)$ and $\mathcal{B}(V, \psi)$ are the unit balls with respect to the norms φ and ψ respectively, and vol is a Haar measure on V. This definition is independent of the choice of Haar measure vol. Moreover, when the norms φ and ψ are Hermitian, it is equal to the number defined in (4). As in the Hermitian case, we let $\widetilde{P}_{\overline{V}}$ be the concave function on $[0, \mathrm{rk}(V)]$ whose graph is the upper boundary of the convex closure of the set of points of the form $(\mathrm{rk}(W), \widehat{\deg}(W))$, where W is a member of the set of subspaces of V. For any $i \in \{1, \ldots, \mathrm{rk}(V)\}$, we let $\widehat{\mu}_i(\overline{V})$ be the slope of the function $\widetilde{P}_{\overline{V}}$ on the interval $[i-1, i]$. The equality (8) holds in this more general context. We let $Z_{\overline{V}}$ be a random variable whose law is an average of Dirac masses at the intermediate slopes $\widehat{\mu}_i(\overline{V})$:

$$\text{the law of } Z_{\overline{V}} \text{ is } \frac{1}{\mathrm{rk}(V)} \sum_{i=1}^{\mathrm{rk}(V)} \delta_{\widehat{\mu}_i(\overline{V})}.$$

We can also introduce a normalised Harder–Narasimhan polygon:

$$P_{\overline{V}}(t) = \frac{1}{\mathrm{rk}(V)} \widetilde{P}_{\overline{V}}(t\,\mathrm{rk}(V)), \quad t \in [0, 1]. \tag{10}$$

Equation (9) holds in this more general setting: we have that

$$P_{\overline{V}}(1) = \mathbb{E}[Z_{\overline{V}}] = \widehat{\mu}(\overline{V}). \tag{11}$$

The following result compares Harder–Narasimhan polygons.

Proposition 2.4. *Let (V, φ, ψ) be an element of $\mathcal{C}$. If ψ' is another norm on V such that*[2] *$\psi' \leqslant \psi$ then we have that $\widetilde{P}_{(V,\varphi,\psi')} \leqslant \widetilde{P}_{(V,\varphi,\psi)}$ as functions on* $[0, \mathrm{rk}(V)]$.

Proof. For any subspace W on V we have that $\mathcal{B}(W, \psi) \subset \mathcal{B}(W, \psi')$ and it follows that $\widehat{\deg}(W, \varphi, \psi) \geq \widehat{\deg}(W, \varphi, \psi')$. The point $(\mathrm{rk}(W), \widehat{\deg}(W, \varphi, \psi'))$ is therefore always below the graph of $\widetilde{P}_{(V,\varphi,\psi)}$. It follows that $\widetilde{P}_{(V,\varphi,\psi')} \leqslant \widetilde{P}_{(V,\varphi,\psi)}$. □

Remark 2.5. Similarly, if (V, φ, ψ) is an element of $\mathcal{C}$ and φ' is another norm on V such that $\varphi' \geqslant \varphi$ then we have that $\widetilde{P}_{(V,\varphi',\psi)} \leqslant \widetilde{P}_{(V,\varphi,\psi)}$ as functions on $[0, \mathrm{rk}(V)]$.

If V is a non-trivial finite dimensional complex vector space and ψ and ψ' are two norms on V then we denote by $d(\psi, \psi')$ the quantity

$$\sup_{0 \neq x \in V} \left| \ln \|x\|_{\psi} - \ln \|x\|_{\psi'} \right|.$$

[2] In other words, for any $x \in V$ we have that $\|x\|_{\psi'} \leqslant \|x\|_{\psi}$.

Corollary 2.6. *Let (V, φ, ψ) be a non-trivial element of $\mathcal{C}$ and let φ' and ψ' be two norms on V. For any $t \in [0, \mathrm{rk}(V)]$ we have that*

$$|\widetilde{P}_{(V,\varphi,\psi)}(t) - \widetilde{P}_{(V,\varphi',\psi')}(t)| \leqslant (d(\varphi,\varphi') + d(\psi,\psi'))\, t.$$

Proof. We denote by ψ_1 and ψ_2 the norms on V such that

$$\|.\|_{\psi_1} = \mathrm{e}^{-d(\psi,\psi')}\|.\|_{\psi} \quad \text{and} \quad \|.\|_{\psi_2} = \mathrm{e}^{d(\psi,\psi')}\|.\|_{\psi}.$$

We have that $\psi_1 \leqslant \psi' \leqslant \psi_2$. Moreover, for any $t \in [0, 1]$ we have that

$$\widetilde{P}_{(V,\varphi,\psi_1)}(t) = \widetilde{P}_{(V,\varphi,\psi)}(t) - d(\psi,\psi')\, t, \quad \widetilde{P}_{(V,\varphi,\psi_1)}(t) = \widetilde{P}_{(V,\varphi,\psi)}(t) + d(\psi,\psi')\, t.$$

By the above proposition we have that

$$|\widetilde{P}_{(V,\varphi,\psi)}(t) - \widetilde{P}_{(V,\varphi,\psi')}(t)| \leqslant d(\psi,\psi')\, t.$$

By the same argument we have that

$$|\widetilde{P}_{(V,\varphi,\psi')}(t) - \widetilde{P}_{(V,\varphi',\psi')}(t)| \leqslant d(\varphi,\varphi')\, t.$$

The sum of these two inequalities gives the required result. □

2.4. John norms

Whilst the elements of $\mathcal{C}$ do not generally satisfy (5), John's ellipsoid technique enables us to show that (5) holds up to an error term. Indeed, if V is a complex vector space of rank $r > 0$ and ϕ is a norm on V then we can find a Hermitian norm ϕ_J which satisfies the following inequalities (cf. [19, p. 84]):

$$\frac{1}{\sqrt{r}}\|.\|_{\phi_J} \leqslant \|.\|_{\phi} \leqslant \|.\|_{\phi_J}.$$

Proposition 2.7. *Let $\overline{V} = (V, \varphi, \psi)$ be a non-trivial element of $\mathcal{C}$. If*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

is a flag of subspaces of V then we have that

$$\left|\widehat{\deg}(V) - \sum_{i=1}^{n} \widehat{\deg}(\overline{V}_i/\overline{V}_{i-1})\right| \leqslant \mathrm{rk}(V)\ln(\mathrm{rk}(V)). \tag{12}$$

Proof. Let r be the rank of V. Consider the object $(V, r^{-1/2}\varphi_J, \psi_J)$, which is an element of $\mathcal{C}^H$. The following equation holds by formula (5):

$$\widehat{\deg}(V, r^{-1/2}\varphi_J, \psi_J) = \sum_{i=1}^{n} \widehat{\deg}(V_i/V_{i-1}, r^{-1/2}\varphi_J, \psi_J).$$

Moreover, since $r^{-1/2}\varphi_J \leqslant \varphi$ and $\psi_J \geqslant \psi$, we have that

$$\widehat{\deg}(V_i/V_{i-1}, \varphi, \psi) \leqslant \widehat{\deg}(V_i/V_{i-1}, r^{-1/2}\varphi_J, \psi_J).$$

It follows that

$$\widehat{\deg}(V, r^{-1/2}\varphi_J, \psi_J) \geqslant \sum_{i=1}^{n} \widehat{\deg}(V_i/V_{i-1}, \varphi, \psi). \tag{13}$$

Moreover, it follows from the relations $\varphi \leqslant \varphi_J$ and $\psi \geqslant r^{-1/2}\psi_J$ that

$$\widehat{\deg}(V, \varphi, \psi) \geqslant \widehat{\deg}(V, r^{-1/2}\varphi_J, \psi_J) - r\ln(r). \tag{14}$$

It follows from the inequalities (13) and (14) that

$$\widehat{\deg}(\overline{V}) \geqslant \sum_{i=1}^{n} \widehat{\deg}(\overline{V}_i/\overline{V}_{i-1}) - r\ln(r).$$

Applying the same argument to $(V, \varphi_J, r^{-1/2}\psi_J)$ we can show that

$$\widehat{\deg}(\overline{V}) \leqslant \sum_{i=1}^{n} \widehat{\deg}(\overline{V}_i/\overline{V}_{i-1}) + r\ln(r).$$

This completes the proof of the proposition. □

2.5. Truncation

Let V be a finite-dimensional complex vector space. If φ is a norm on V and a is a real number then we let $\varphi(a)$ be the norm on V such that

$$\forall\, x \in V, \quad \|x\|_{\varphi(a)} = \mathrm{e}^{a}\|x\|_{\varphi}.$$

If φ and ψ are two norms on V then we denote by $\varphi \vee \psi$ the norm on V such that

$$\forall\, x \in V, \quad \|x\|_{\varphi\vee\psi} = \max(\|x\|_{\varphi}, \|x\|_{\psi}).$$

Proposition 2.8. *Let $\overline{V} = (V, \varphi, \psi)$ be a non-zero element of $\mathcal{C}$ and let a be a real number. We have that*

$$\left|\widehat{\deg}(V, \varphi, \psi \vee \varphi(a)) - \sum_{i=1}^{\mathrm{rk}(V)} \max(\widehat{\mu}_i(\overline{V}), a)\right| \leqslant 2\,\mathrm{rk}(V)\ln(\mathrm{rk}(V)) + \frac{\mathrm{rk}(V)}{2}\ln(2).$$

Proof. Let r be the dimension of V. We will first prove the following equation.

$$\sum_{i=1}^{r} \max(\widehat{\mu}_i(\overline{V}), a) = \sup_{t\in[0,r]} \Big(\widetilde{P}_{\overline{V}}(t) - at\Big) + ar. \tag{15}$$

As the function $\widetilde{P}_{\overline{V}}(t) - at$ is affine on each segment $[i-1, i]$ ($i \in \{1, \ldots, r\}$) we get that

$$\sup_{t\in[0,r]} \big(\widetilde{P}_{\overline{V}}(t) - at\big) = \max_{i\in\{0,\ldots,r\}} \Bigg(\sum_{1\leqslant j\leqslant i} \big(\widehat{\mu}_i(\overline{V}) - a\big)\Bigg) = \sum_{i=1}^{r} \max\big(\widehat{\mu}_i(\overline{V}) - a, 0\big).$$

It follows that

$$\sup_{t\in[0,r]}\big(\widetilde{P}_{\overline{V}}(t)-at\big)+ar=\sum_{i=1}^{r}\Big(\max\big(\widehat{\mu}_i(\overline{V})-a,0\big)+a\Big)=\sum_{i=1}^{r}\max(\widehat{\mu}_i(\overline{V}),a).$$

We start by proving the proposition in the special case where φ and ψ are Hermitian. There is then a basis $\boldsymbol{e}=(e_i)_{i=1}^r$ which is orthonormal for φ and orthogonal for ψ. For any $i\in\{1,\ldots,r\}$ set $\lambda_i=\|e_i\|_\psi$. Without loss of generality, we may assume that $\lambda_1\geqslant\cdots\geqslant\lambda_r$. We therefore have that $\widehat{\mu}_i(\overline{V})=\ln(\lambda_i)$ for any $i\in\{1,\ldots,r\}$. Let ψ' be the Hermitian norm on V such that

$$\|x_1e_1+\cdots+x_re_r\|_{\psi'}^2=\sum_{i=1}^{r}x_i^2\max(\lambda_i^2,\mathrm{e}^{2a}).$$

We then have that

$$|x_1e_1+\cdots+x_re_r\|_{\psi\vee\varphi(a)}^2=\max\left(\sum_{i=1}^{r}x_i^2\lambda_i^2,\sum_{i=1}^{r}x_i^2\mathrm{e}^{2a}\right).$$

and it follows that

$$\|.\|_{\psi\vee\varphi(a)}\leqslant\|.\|_{\psi'}\leqslant\sqrt{2}\|.\|_{\psi\vee\varphi(a)}.$$

It follows that

$$\widehat{\deg}(V,\varphi,\psi\vee\varphi(a))\leqslant\widehat{\deg}(V,\varphi,\psi')\leqslant\frac{r}{2}\log(2)+\widehat{\deg}(V,\varphi,\psi\vee\varphi(a)),$$

and hence

$$\left|\widehat{\deg}(V,\varphi,\psi\vee\varphi(a))-\sum_{i=1}^{r}\max(\widehat{\mu}_i(\overline{V}),a)\right|\leqslant\frac{r}{2}\log(2).$$

We now deal with the general case. We choose Hermitian norms φ_1 and ψ_1 such that $d(\varphi,\varphi_1)\leqslant\frac{1}{2}\log(r)$ and $d(\psi,\psi_1)\leqslant\frac{1}{2}\log(r)$. Applying the above result to (V,φ_1,ψ_1) we get that

$$\left|\deg(V,\varphi_1,\psi_1\vee\varphi_1(a))-\sum_{i=1}^{r}\max(\widehat{\mu}_i(V,\varphi_1,\psi_1),a)\right|\leqslant\frac{r}{2}\log(2).$$

Moreover, we have that

$$d(\psi\vee\varphi(a),\psi_1\vee\varphi_1(a))\leqslant\max\big(d(\varphi,\varphi_1),d(\psi,\psi_1)\big)$$

and hence

$$\begin{aligned}|\deg(V,\varphi,\psi\vee\varphi(a))-\deg(V,\varphi_1,\psi_1\vee\varphi_1(a))|&\leqslant d(\varphi,\varphi_1)r\\&+\max(d(\varphi,\varphi_1),d(\psi,\psi_1))r,\end{aligned}$$

which is bounded above by $r\ln(r)$. Moreover, by 2.7 we get that

$$\left|\widetilde{P}_{\overline{V}}(t)-\widetilde{P}_{(V,\varphi_1,\psi_1)}(t)\right|\leqslant\big(d(\varphi,\varphi_1)+d(\psi,\psi_1)\big)t\leqslant t\ln(r)\leqslant r\ln(r),$$

and hence

$$\left| \sup_{t\in[0,r]} \left(\widetilde{P}_{\overline{V}}(t) - at\right) - \sup_{t\in[0,r]} \left(\widetilde{P}_{(V,\varphi_1,\psi_1)}(t) - at\right)\right| \leqslant r\ln(r).$$

By (15) we get that

$$\left|\widehat{\deg}(V,\varphi,\psi\vee\varphi(a)) - \sum_{i=1}^{r}\max(\widehat{\mu}_i(\overline{V}),a)\right| \leqslant 2r\ln(r) + \frac{r}{2}\log(2).$$

□

3. The non-Archimedean analogue

In this section, we develop an analogue for slopes for finite dimensional vector spaces over a non-archimedean field k equipped with two ultrametric norms. Let k be a field equipped with a complete non-archimedean absolute value function $|.|$.

3.1. Ultrametric norms

Let V be a k-vector space of dimension r equipped with a norm $\|.\|$. As k is assumed to be complete the topology on V is induced by any isomorphism $k^r \to V$. (We refer the reader to [8, I.§2, no. 3] theorem 2 and the remark on page I.15 for a proof of this fact). In particular, any subspace of V is closed (cf. *loc. cit.* corollary 1 of theorem 2).

Let $(V,\|.\|)$ be a finite-dimensional ultra-normed vector space on k. A basis $\boldsymbol{e}=(e_i)_{i=1}^r$ of V is said to be *orthogonal* if the following holds:

$$\forall\,(\lambda_1,\ldots,\lambda_r)\in k^r,\quad \|\lambda_1e_1+\cdots+\lambda_re_r\| = \max_{i\in\{1,\ldots,r\}}|\lambda_i|\cdot\|e_i\|.$$

An ultra-normed vector space does not necessarily have an orthogonal basis, but the following proposition shows that an asymptotic version of Gram-Schmidt's algorithm is still valid in this context.

Proposition 3.1. *Let $(V,\|.\|)$ be a ultra-normed k-vector space of dimension $r\geqslant 1$. Let*

$$0=V_0\subsetneq V_1\subsetneq V_2\subsetneq\cdots\subsetneq V_r=V$$

be a full flag of subspaces of V. For any $\varepsilon\in\,]0,1[$ there is a basis $\boldsymbol{e}=(e_i)_{i=1}^r$ compatible with the flag[3] *such that*

$$\forall\,(\lambda_1,\ldots,\lambda_r)\in k^r,\quad \|\lambda_1e_1+\cdots+\lambda_re_r\| \geqslant (1-\varepsilon)\max_{i\in\{1,\ldots,r\}}|\lambda_i|\cdot\|e_i\|. \tag{16}$$

[3] We say that a basis $\boldsymbol{e}$ is compatible with a full flag $0=V_0\subsetneq V_1\subsetneq V_2\subsetneq\cdots\subsetneq V_r=V$ if for any $i\in\{1,\ldots,r\}$, we have that $\mathrm{card}(V_i\cap\boldsymbol{e})=i$.

Proof. We proceed by induction on r, the dimension of V. The case $r = 1$ is trivial. Assume the proposition holds for all spaces of dimension $< r$ for some $r \geqslant 2$. Applying the induction hypothesis to V_{r-1} and the flag $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{r-1}$ we get a basis $(e_1, \ldots, e_{r-1})$ compatible with the flag such that

$$\forall\,(\lambda_1, \ldots, \lambda_{r-1}) \in k^{r-1}, \quad \|\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1}\| \geqslant (1-\varepsilon) \max_{i \in \{1,\ldots,r-1\}} |\lambda_i| \cdot \|e_i\|. \tag{17}$$

Let x be an element of $V \setminus V_{r-1}$ and let y be a point in V_{r-1} such that

$$\|x - y\| \leqslant (1 - \varepsilon)^{-1} \mathrm{dist}(x, V_{r-1}). \tag{18}$$

(The distance between x and V_{r-1} is strictly positive because V_{r-1} is closed in V.) We choose $e_r = x - y$. The basis $e_1, \ldots, e_r$ is compatible with the flag $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$. Let $(\lambda_1, \ldots, \lambda_r)$ be an element of k^r: we wish to find a lower bound for the norm of $z = \lambda_1 e_1 + \cdots + \lambda_r e_r$. By (18) we have that

$$\|z\| \geqslant |\lambda_r| \cdot \mathrm{dist}(x, V_{r-1}) \geqslant (1 - \varepsilon)|\lambda_r| \cdot \|e_r\|.$$

This provides our lower bound when $\|\lambda_r e_r\| \geqslant \|\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1}\|$. If $\|\lambda_r e_r\| < \|\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1}\|$ then we have $\|z\| = \|\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1}\|$ because the norm is ultrametric. By the induction hypothesis (17) we have that $\|z\| \geqslant (1 - \varepsilon)|\lambda_i| \cdot \|e_i\|$ for any $i \in \{1, \ldots, r - 1\}$. This completes the proof of the proposition. □

Remark 3.2. If k is locally compact then the above equation holds for $\varepsilon = 0$ since the distance appearing in (18) is then attained.

In order to make the computations in Sect. 3.2 clearer, we use the notion of α-orthogonality of a basis in an ultra-normed k-vector space of finite (see [18, §2.3]). Note that this notion has also been used by Gaudron in the study of linear forms of logarithms, see [14, §3.5].

Definition 3.3. Let V be an ultra-normed k-vector space of finite rank and α a real number in $]0, 1]$. We say that a basis $\boldsymbol{e} = (e_1, \ldots, e_r)$ of V is *α-orthogonal* if for any $(\lambda_1, \ldots, \lambda_r) \in k^r$ we have that

$$\|\lambda_1 e_1 + \cdots + \lambda_r e_r\| \geqslant \alpha \max(|\lambda_1| \cdot \|e_1\|, \ldots, |\lambda_r| \cdot \|e_r\|).$$

By definition, 1-orthogonality is the same thing as orthogonality.

For any ultra-normed finite dimensional k-vector space V we let $V^\vee = \mathrm{Hom}_k(V, k)$ be its dual space with the operator norm. This is also a finite-dimensional ultra-normed k-vector space.

Let (V_1, φ_1) and (V_2, φ_2) be two ultra-normed finite-dimensional k-vector spaces. We can then identify $V_1 \otimes V_2$ with $\mathrm{Hom}_k(V_1^\vee, V_2)$ and equip it with the operator norm. This ultrametric norm on $V_1 \otimes V_2$, is called the *tensor* norm and is denoted by $\varphi_1 \otimes \varphi_2$. We can construct a tensor norm for a tensor product of multiple ultra-normed spaces recursively (cf. [13, remarque 3.8]).

The following classical property of α-orthogonality for tensorial ultranormed vector space will be useful in Sect. 3.2 for the Harder–Narasimhan formalism of vector spaces equipped with two ultranorms. We refer the readers to [18, Corollary 10.2.10] for a proof.

Proposition 3.4. *Let V and W be two finite dimensional ultra-normed k-vector spaces and let α be a real number in $]0,1]$. If $(s_i)_{i=1}^n$ and $(t_j)_{j=1}^m$ are α-orthogonal bases of V and W respectively then $(s_i \otimes t_j)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}}$ is an α^2-orthogonal basis of $V \otimes W$.*

For any k-vector space V of finite dimension r equipped with an ultrametric norm $\|.\|$ we equip $\det(V) = \Lambda^r(V)$ with the quotient norm induced by the canonical map $V^{\otimes r} \to \Lambda^r(V)$. The ultrametric Hadamard inequality implies that for any basis $(e_1, \dots, e_r)$ of V we have that

$$\|e_1 \wedge \cdots \wedge e_r\| \leqslant \|e_1 \otimes \cdots \otimes e_r\| = \prod_{i=1}^{r} \|e_i\|. \tag{19}$$

Equality holds when the basis $(e_1, \dots, e_r)$ is orthogonal. If the basis is α-orthogonal then we have that

$$\|e_1 \wedge \cdots \wedge e_r\| \geqslant \alpha^r \prod_{i=1}^{r} \|e_i\|. \tag{20}$$

This follows from[4] Proposition 3.4.

3.2. *Arakelov degree and the Harder–Narasimhan polygon*

Let $\mathcal{C}_k$ be the class of finite dimensional k-vector spaces equipped with two ultrametric norms. If $\overline{V} = (V, \|.\|_\varphi, \|.\|_\psi)$ is an element of $\mathcal{C}_k$ we let $\widehat{\deg}(\overline{V})$ be the following number

$$-\ln \|s_1 \wedge \cdots \wedge s_r\|_\varphi + \ln \|s_1 \wedge \cdots \wedge s_r\|_\psi,$$

where $(s_1, \dots, s_r)$ is an arbitrary k-basis of V. This construction does not depend on the choice of $(s_1, \dots, s_r)$ by the trivial product formula

$$\forall\, a \in k^\times, \quad -\ln|a| + \ln|a| = 0.$$

If V is non-trivial we let $\widehat{\mu}(\overline{V})$ be the quotient $\widehat{\deg}(\overline{V})/\operatorname{rk}(V)$. It follows from Proposition 3.1, Hadamard's inequality (19) and the inverse Hadamard inequality (20) that for any flag of subspaces of V

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

---

[4] By induction, Proposition 3.4 implies that $(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(r)})_{1\leqslant\sigma(1),\dots,\sigma(r)\leqslant r}$ is an α^r-basis of $V^{\otimes r}$. If $\xi = \sum_\sigma a_\sigma e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(r)}$ is an element of $V^{\otimes n}$ whose image in $\Lambda^r V$ is $e_1 \wedge \cdots \wedge e_r$ then $\sum_{\sigma\in\mathfrak{S}_r} a_\sigma (-1)^{\operatorname{sgn}(\sigma)} = 1$ where $\mathfrak{S}_r$ is the r-th symmetric group. There is at least one $\sigma \in \mathfrak{S}_r$ such that $|a_\sigma| \geqslant 1$. It follows that $\|\xi\| \geqslant \alpha^r \prod_{i=1}^r \|e_i\|$.

we have that

$$\widehat{\deg}(\overline{V}) = \sum_{i=1}^{n} \widehat{\deg}(\overline{V}_i/\overline{V}_{i-1}), \tag{21}$$

using the subquotient norms.

We let $\widetilde{P}_{\overline{V}}$ be the function on $[0, r]$ whose graph is the concave upper bound of the set of points in $\mathbb{R}^2$ of the form $(\mathrm{rk}(W), \widehat{\deg}(\overline{W}))$, where W runs over the set of subspaces of V. This is a concave function which is affine on each piece $[i-1, i]$ ($i \in \{1, \dots, r\}$). For any $i \in \{1, \dots, r\}$ we let $\widehat{\mu}_i(\overline{V})$ be the slope of this function on $[i-1, i]$. We introduce a normalised version of $\widetilde{P}_{\overline{V}}$ by setting

$$\forall\, t \in [0, 1], \quad P_{\overline{V}}(t) = \frac{1}{\mathrm{rk}(V)} \widetilde{P}_{\overline{V}}(t\,\mathrm{rk}(V)).$$

Let $Z_{\overline{V}}$ be a random variable whose probability law is given by

$$\frac{1}{\mathrm{rk}(V)} \sum_{i=1}^{\mathrm{rk}(V)} \delta_{\widehat{\mu}_i(\overline{V})}.$$

The following equality also holds in the non-archimedean case

$$\widehat{\mu}(\overline{V}) = P_{\overline{V}}(1) = \mathbb{E}[Z_{\overline{V}}]. \tag{22}$$

The results of Sect. 2.3—in particular Proposition 2.4 and Corollary 2.6—still hold for members of $\mathcal{C}_k$ (and their proofs are similar). We now summarise these properties:

Proposition 3.5. *Let (V, φ, ψ) be an element of $\mathcal{C}_k$. Let φ' and ψ' be two ultrametric norms on V.*

(1) If $\psi' \leqslant \psi$ then $\widetilde{P}_{(V,\varphi,\psi')} \leqslant \widetilde{P}_{(V,\varphi,\psi)}$.
(2) If $\varphi' \geqslant \varphi$ then $\widetilde{P}_{(V,\varphi',\psi)} \leqslant \widetilde{P}_{(V,\varphi',\psi)}$.
(3) In general we have that

$$\forall\, t \in [0, \mathrm{rk}(V)], \quad \left|\widetilde{P}_{(V,\varphi,\psi)}(t) - \widetilde{P}_{(V,\varphi',\psi')}(t)\right| \leqslant \big(d(\varphi, \varphi') + d(\psi, \psi')\big)t.$$

The following proposition can be seen as an ultrametric analogue of the Courant–Fischer theorem.

Proposition 3.6. *Let $\overline{V} = (V, \varphi, \psi)$ be an element of $\mathcal{C}_k$ of dimension $r \geqslant 1$. For any $i \in \{1, \dots, r\}$ we have that*

$$\widetilde{P}_{\overline{V}}(i) = \sup_{\substack{W \subset V \\ \mathrm{rk}(W)=i}} \widehat{\deg}(\overline{W}). \tag{23}$$

Proof. We prove by induction on r that for any $\alpha \in]0,1[$ there is a basis in V which is α-orthogonal for both φ and ψ. The case $r=1$ is trivial. Suppose that this statement has been proved for all elements of $\mathcal{C}_k$ of dimension $< r$. We choose $e_1 \in V \setminus \{0\}$ such that

$$\frac{\|e_1\|_\psi}{\|e_1\|_\varphi} \leqslant (\sqrt[4]{\alpha})^{-1} \inf_{0\neq x\in V} \frac{\|x\|_\psi}{\|x\|_\varphi}. \tag{24}$$

By Proposition 3.1 there is a subspace $W \subset V$ of dimension $r-1$ such that

$$\forall\, y \in W, \quad \|e_1 + y\|_\varphi \geqslant \sqrt[4]{\alpha} \max(\|e_1\|_\varphi, \|y\|_\varphi). \tag{25}$$

By (24) and (25) we have that

$$\forall\, y \in W, \quad \|e_1 + y\|_\psi \geqslant \sqrt[4]{\alpha} \cdot \|e_1 + y\|_\varphi \cdot \frac{\|e_1\|_\psi}{\|e_1\|_\varphi} \geqslant \sqrt{\alpha} \cdot \|e_1\|_\psi.$$

Moreover, as ψ is ultrametric, if $\|y\|_\psi > \|e_1\|_\psi$ then $\|e_1+y\|_\psi = \|y\|_\psi$. It follows that

$$\|e_1 + y\|_\psi \geqslant \sqrt{\alpha} \max(\|e_1\|_\psi, \|y\|_\psi). \tag{26}$$

By the induction hypothesis, there is a basis $(e_2,\ldots,e_r)$ of W which is $\sqrt{\alpha}$-orthogonal for both φ and ψ. Let us prove that $(e_1,\ldots,e_r)$ is α-orthogonal for both φ and ψ. Let $(\lambda_1,\ldots,\lambda_r)$ be an element of k^r with $\lambda_1 \neq 0$. By (25) we have that

$$\begin{aligned}
\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\varphi &\geqslant \sqrt[4]{\alpha} \cdot \max(|\lambda_1| \cdot \|e_1\|_\varphi, \|\lambda_2 e_2 + \cdots + \lambda_r e_r\|_\varphi) \\
&\geqslant \sqrt[4]{\alpha} \cdot \max(|\lambda_1| \cdot \|e_1\|_\varphi, \sqrt{\alpha} \cdot \max(|\lambda_2| \cdot \|e_2\|_\varphi, \ldots, |\lambda_r| \cdot \|e_r\|_\varphi)) \\
&\geqslant \alpha \max(|\lambda_1| \cdot \|e_1\|_\varphi, \ldots, |\lambda_r| \cdot \|e_r\|_\varphi),
\end{aligned}$$

where the second inequality comes from the fact that $(e_2,\ldots,e_r)$ is $\sqrt{\alpha}$-orthogonal for W for the norm $\|.\|_\varphi$. This inequality also holds when $\lambda_1 = 0$ (here we use directly the fact that $(e_2,\ldots,e_r)$ is an $\sqrt{\alpha}$-orthogonal basis). Similarly, it follows from (26) that $(e_1,\ldots,e_r)$ is an α-orthogonal basis for V with respect to $\|.\|_\psi$.

We now prove the proposition. We deal first with the case where there is a basis $(s_1,\ldots,s_r)$ which is orthogonal for both φ and ψ. For any $i \in \{1,\ldots,r\}$ let a_i be the logarithm of the ratio $\|s_i\|_\psi/\|s_i\|_\varphi$. Without loss of generality we may assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$. For any integer $m \in \{1,\ldots,r\}$ the vectors $s_{i_1} \wedge \cdots \wedge s_{i_m}$ $(1 \leqslant i_1 < \cdots < i_m \leqslant r)$ form a basis for $\Lambda^m(V)$ which is orthogonal for both φ and ψ. For any m-dimensional subspace $W \subset V$, writing a non-zero element of $\Lambda^m W$ as a sum of elements of the form $s_{i_1} \wedge \cdots \wedge s_{i_m}$ enables us to prove that

$$\widehat{\deg}(\overline{W}) \leqslant a_1 + \cdots + a_m,$$

and equality is achieved when W is generated by the $s_1,\ldots,s_m$.

In the general case, the above proposition enables us to construct a sequence of objects (V, φ_n, ψ_n) $(n \in \mathbb{N})$ in $\mathcal{C}_k$ such that[5]

$$\lim_{n\to+\infty} d(\varphi_n, \varphi) + d(\psi_n, \psi) = 0$$

and for any n there is a basis of V which is orthogonal for both φ_n and ψ_n. On the one hand,

$$\widetilde{P}_{(V,\varphi_n,\psi_n)}(i) = \sup_{\substack{W\subset V\\ \mathrm{rk}(W)=i}} \widehat{\deg}(W, \varphi_n, \psi_n),$$

and on the other, we have that

$$\left|\widetilde{P}_{(V,\varphi_n,\psi_n)}(t) - \widetilde{P}_{(V,\varphi,\psi)}(t)\right| \leqslant (d(\varphi_n, \varphi) + d(\psi_n, \psi))t,$$

and for any subspace $W \subset V$ we have that

$$\left|\widehat{\deg}(W, \varphi_n, \psi_n) - \widehat{\deg}(W, \varphi, \psi)\right| \leqslant (d(\varphi_n, \varphi) + d(\psi_n, \psi))\,\mathrm{rk}(W).$$

As $n \to +\infty$ we obtain the desired result. □

3.3. Truncation

The results in Sect. 2.5 are still valid for elements of $\mathcal{C}_k$. The proof is simpler because we only consider ultrametric norms. Let V be a finite dimensional k-vector space and let φ be an ultrametric norm on V. For any real number a let $\varphi(a)$ be the norm on V such that

$$\forall\, x \in V, \quad \|x\|_{\varphi(a)} = \mathrm{e}^a \|x\|_\varphi.$$

If φ and ψ are two ultrametric norms on V we let $\varphi \vee \psi$ be the norm on V such that

$$\forall\, x \in V, \quad \|x\|_{\varphi\vee\psi} = \max(\|x\|_\varphi, \|x\|_\psi).$$

This is also an ultrametric norm on V.

Proposition 3.7. *Let $\overline{V} = (V, \varphi, \psi)$ be a non-trivial element of $\mathcal{C}_k$ and let a be a real number. We have that*

$$\widehat{\deg}(V, \varphi, \psi \vee \varphi(a)) = \sum_{i=1}^{\mathrm{rk}(V)} \max(\widehat{\mu}_i(\overline{V}), a)$$

[5] As in the complex case, for any pair (η, η') of ultrametric norms on V we set

$$d(\eta, \eta') = \sup_{0\neq x\in V} \left|\ln \|x\|_\eta - \ln \|x\|_{\eta'}\right|.$$

Proof. We start with the case where V has a basis $(e_1, \ldots, e_r)$ which is orthogonal for both φ and ψ. Without loss of generality, we have that

$$\ln \frac{\|e_i\|_\psi}{\|e_i\|_\varphi} = \widehat{\mu}_i(\overline{V}).$$

The basis $(e_1, \ldots, e_r)$ is also orthogonal for the norm $\psi \vee \varphi(a)$ and we have that

$$\ln \frac{\|e_i\|_\psi}{\|e_i\|_\varphi} = \max(\widehat{\mu}_i(\overline{V}), a).$$

The result follows. In general, we can approximate (φ, ψ) by pairs of norms $((\varphi_n, \psi_n))_{n \geqslant 1}$ such that for every n the vector space V has a basis orthogonal for both φ_n and ψ_n (cf. the proof of Proposition 3.6). Passing to the limit $n \to +\infty$, we get our result. □

4. Equilibrium energy

In this section we fix a field k, which is either $\mathbb{C}$ with the usual absolute value, or a complete field equipped with a non-archimedean absolute value. If $k = \mathbb{C}$ then we will denote by $\mathcal{C}_k$ the class $\mathcal{C}$ defined in Sect. 2.3.

4.1. Monomial bases and the Okounkov semi-group

In this subsection we recall the construction of the Okounkov semi-group of a graded linear series. We refer the reader to [6,15–17] for more details.

Consider an integral projective scheme X of dimension $d \geqslant 1$ defined over the field k. We assume that the scheme X has a regular rational point x: the local ring $\mathcal{O}_{X,x}$ is then a regular local ring of dimension d. We fix a regular sequence $(z_1, \ldots, z_d)$ in its maximal ideal $\mathfrak{m}_x$. The formal completion of $\mathcal{O}_{X,x}$ with respect to the maximal ideal $\mathfrak{m}_x$ is isomorphic to the algebra of formal series in the parameters $z_1, \ldots, z_d$ (cf. [12, Proposition 10.16]).

If we choose a monomial ordering[6] $\leqslant$ on $\mathbb{N}^d$ we obtain a decreasing $\mathbb{N}^d$-filtration (called the *Okounkov filtration*) $\mathcal{F}$ on $\widehat{\mathcal{O}}_{X,x}$ such that $\mathcal{F}^\alpha(\widehat{\mathcal{O}}_{X,x})$ is the ideal generated by monomials of the form z^β such that $\beta \geqslant \alpha$. This filtration is multiplicative: we have that

$$\mathcal{F}^\alpha(\widehat{\mathcal{O}}_{X,x})\mathcal{F}^\beta(\widehat{\mathcal{O}}_{X,x}) \subset \mathcal{F}^{\alpha+\beta}(\widehat{\mathcal{O}}_{X,x}).$$

The filtration $\mathcal{F}$ induces by grading an $\mathbb{N}^d$-graded algebra $\mathrm{gr}(\mathcal{O}_{X,x})$ which is isomorphic[7] to $k[z_1, \ldots, z_d]$. In particular, for any $\alpha \in \mathbb{N}^d$, $\mathrm{gr}^\alpha(\mathcal{O}_{X,x})$ is a rank-one vector space on k.

[6] This is a total order $\leqslant$ on $\mathbb{N}^d$ such that $0 \leqslant \alpha$ for any $\alpha \in \mathbb{N}^d$ and $\alpha \leqslant \alpha'$ implies $\alpha + \beta \leqslant \alpha' + \beta$ for any α, α' and β in $\mathbb{N}^d$.

[7] This follows from the fact that $\mathcal{O}_{X,x}$ is dense in $\widehat{\mathcal{O}}_{X,x}$.

If L is an inversible $\mathcal{O}_X$-module then on taking a local trivialisation in a neighbourhood of x we can identify L_x with $\mathcal{O}_{X,x}$. The filtration $\mathcal{F}$ then induces a decreasing $\mathbb{N}^d$-filtration on $H^0(X, L)$ which is independent of the choice of trivialisation. For any $s \in H^0(X, L)$ we denote by $\mathrm{ord}(s)$ the upper bound of the set of $\alpha \in \mathbb{N}^d$ such that $s \in \mathcal{F}^\alpha H^0(X, L)$. We have that

$$\forall\, s, s' \in H^0(X, L), \quad \mathrm{ord}(s + s') \geqslant \min\big(\mathrm{ord}(s), \mathrm{ord}(s')\big).$$

Moreover, for any $s \in H^0(X, L)$ and any $a \in k^\times$ we have that $\mathrm{ord}(s) = \mathrm{ord}(as)$.

Let L be an invertible $\mathcal{O}_X$-module. We let $V_\bullet(L)$ be the graded ring $\bigoplus_{n\geqslant 0} H^0(X, nL)$ (with the additive notation for the tensor product of invertible sheaves). By a *graded linear system* of L we mean a graded subalgebra of $V_\bullet(L)$. Any graded linear system $V_\bullet$ of L can be identified, on choosing a local trivialisation of L around x, with a graded subalgebra of the algebra of polynomials $\mathcal{O}_{X,x}[T]$. The filtration $\mathcal{F}$ induces a decreasing $\mathbb{N}^d$-filtration on each homogeneous piece V_n. We denote by $\mathrm{gr}(V_\bullet)$ the $\mathbb{N}^{d+1}$-graded k-algebra induced by this filtration. This is an $\mathbb{N}^{d+1}$-graded subalgebra of $\mathrm{gr}(\mathcal{O}_{X,x})[T] \cong k[z_1, \ldots, z_d, T]$. In particular, the elements $(n, \alpha) \in \mathbb{N}^{d+1}$ such that $\mathrm{gr}^{(n,\alpha)}(V_\bullet) \neq \{0\}$ form a sub-semigroup of $\mathbb{N}^{d+1}$ which we denote by $\Gamma(V_\bullet)$. For any $n \in \mathbb{N}$ we denote by $\Gamma(V_n)$ the subset of $\mathbb{N}^d$ of elements α such that $(n, \alpha) \in \Gamma(V_\bullet)$.

4.2. Monomial norms

As above, we consider an integral projective scheme X of dimension $d \geqslant 1$ over $\mathrm{Spec}\, k$. We fix a regular rational point $x \in X(k)$ (it is assumed that such a point exists), a system of parameters $z = (z_1, \ldots, z_d)$ at x and a monomial order on $\mathbb{N}^d$. Let L be an invertible $\mathcal{O}_X$-module and let $V_\bullet$ be a graded linear system of L. We assume that every k vector space V_n is equipped with two norms φ_n and ψ_n, which are ultrametric if k is non-archimedean. We assume moreover that these norms are submultiplicative—i.e. that for any $(n, m) \in \mathbb{N}^2$ and any $(s_n, s_m) \in V_n \times V_m$ we have that

$$\|s_n \otimes s_m\|_{\varphi_{n+m}} \leqslant \|s_n\|_{\varphi_n} \cdot \|s_m\|_{\varphi_m}, \quad \|s_n \otimes s_m\|_{\psi_{n+m}} \leqslant \|s_n\|_{\psi_n} \cdot \|s_m\|_{\psi_m}. \quad (27)$$

In this subsection, we study the asymptotic behaviour of $\widehat{\deg}(V_n, \varphi_n, \psi_n)$. As the Harder–Narasimhan filtration is not functorial in $\mathcal{C}_k$ we cannot study this problem directly using the method developped in [7]. We will avoid this problem by using Okounkov filtrations. The norms φ_n and ψ_n induce quotient norms on each of the sub-quotients $\mathrm{gr}^\alpha(V_n)$ ($\alpha \in \Gamma(V_n)$) which we denote by φ_n^α and ψ_n^α respectively. By the results of previous sections, notably (12) and (21), we have that

$$\left|\widehat{\deg}(V_n, \varphi_n, \psi_n) - \sum_{\alpha\in\Gamma(V_n)} \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n^\alpha, \psi_n^\alpha)\right| \leqslant A_k(\mathrm{rk}(V_n)),$$

where $A_k(r) = r\ln(r)$ if $k = \mathbb{C}$ and $A_k(r) = 0$ if k is non-archimedean. We deduce that

$$\lim_{n\to+\infty}\left|\frac{\widehat{\mu}(V_n, \varphi_n, \psi_n)}{n} - \frac{1}{n\#\Gamma(V_n)} \sum_{\alpha\in\Gamma(V_n)} \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n^\alpha, \psi_n^\alpha)\right| = 0. \quad (28)$$

We denote by $(\mathrm{gr}(V_n), \widehat{\varphi}_n)$ and $(\mathrm{gr}(V_n), \widehat{\psi}_n)$ the orthogonal direct sum of $(\mathrm{gr}^\alpha(V_n), \varphi_n^\alpha)_{\alpha\in\Gamma(V_n)}$ and $(\mathrm{gr}^\alpha(V_n), \psi_n^\alpha)_{\alpha\in\Gamma(V_n)}$ respectively. Then we have that

$$\sum_{\alpha\in\Gamma(V_n)} \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \psi_n) = \widehat{\deg}(\mathrm{gr}(V_n), \widehat{\varphi}_n, \widehat{\psi}_n).$$

As the semi-group $\Gamma(V_\bullet)$ is a multiplicative basis for the algebra

$$k[\Gamma(V_\bullet)] \cong \bigoplus_{n\geqslant 0} \mathrm{gr}(V_n)$$

we can construct a new norm η_n on each space $\mathrm{gr}(V_n)$ as follows. For any $\gamma \in \Gamma(V_\bullet)$ we let s_γ be the canonical image of $\gamma \in \Gamma(V_\bullet)$ in the algebra $k[\Gamma(V_\bullet)]$ and we equip

$$\mathrm{gr}(V_n) = \bigoplus_{\alpha\in\Gamma(V_n)} ks_{(n,\alpha)}$$

with the norm η_n such that the vectors $s_{n,\alpha}$ are orthogonal of norm 1. Using these auxiliary norms we can write the degree $\widehat{\deg}(\mathrm{gr}(V_n), \widehat{\varphi}_n, \widehat{\psi}_n)$ as a difference

$$\widehat{\deg}(\mathrm{gr}(V_n), \widehat{\varphi}_n, \eta_n) - \widehat{\deg}(\mathrm{gr}(V_n), \widehat{\psi}_n, \eta_n),$$

or alternatively

$$\sum_{\alpha\in\Gamma(V_n)} \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \eta_n) - \sum_{\alpha\in\Gamma(V_n)} \widehat{\deg};\, (\mathrm{gr}^\alpha(V_n), \psi_n, \eta_n).$$

It is easy to see that the real valued functions $(n, \alpha) \mapsto \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \eta_n)$ and $(n, \alpha) \mapsto \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \eta_n)$ defined on $\Gamma(V_\bullet)$ are superadditive, so their asymptotic behaviour can be studied using the methods developped in [7].

4.3. Limit theorem

In this subsection we fix an integer $d \geqslant 1$ and a sub-semigroup Γ in $\mathbb{N}^{d+1}$. For any integer $n \in \mathbb{N}$ we denote by Γ_n the set $\{\alpha \in \mathbb{N}^d \mid (n, \alpha) \in \Gamma\}$. We suppose that the semi-group Γ verifies the following conditions (cf. [16, §2.1]):

(a) $\Gamma_0 = \{\mathbf{0}\}$,
(b) there is a finite subset B in $\{1\} \times \mathbb{N}^d$ such that Γ is contained in the sub-monoid of $\mathbb{N}^{d+1}$ generated by B,
(c) the group $\mathbb{Z}^{d+1}$ is generated by Γ.

We let $\Sigma(\Gamma)$ be the (closed) convex cone in $\mathbb{R}^{d+1}$ generated by Γ. Under the above conditions the projection of $\Sigma \cap (\{1\} \times \mathbb{R}^d)$ into $\mathbb{R}^d$ is a convex body in $\mathbb{R}^d$, denoted $\Delta(\Gamma)$. Moreover, we have that

$$\lim_{n\to+\infty} \frac{\#\Gamma_n}{n^d} = \mathrm{vol}(\Delta(\Gamma)),$$

where vol(.) is Lesbesgue measure on $\mathbb{R}^d$ (cf. [16, proposition 2.1]).

We say that a function $\Phi : \Gamma \to \mathbb{R}$ is *superadditive* if $\Phi(\gamma + \gamma') \geqslant \Phi(\gamma) + \Phi(\gamma')$. In what follows, we study the asymptotic properties of super-additive functions.

Lemma 4.1. *Let Φ be a superadditive function defined on Γ such that $\Phi(0,0)=0$.*

(1) For any real number t the set $\Gamma_\Phi^t := \{(n,\alpha)\in\Gamma \mid \Phi(n,\alpha)\geqslant nt\}$ is a sub-semigroup of Γ.
(2) If $t\in\mathbb{R}$ is a real number such that

$$t<\lim_{n\to+\infty}\sup_{\alpha\in\Gamma_n}\frac{1}{n}\Phi(n,\alpha),$$

then Γ_Φ^t satisfies conditions (a)–(c) above.

Proof. (1) As Φ is superadditive, for any (n,α) and (m,β) in Γ_Φ^t we have that

$$\Phi(n+m,\alpha+\beta)\geqslant\Phi(n,\alpha)+\Phi(m,\beta)\geqslant nt+mt=(n+m)t,$$

and hence $(n+m,\alpha+\beta)\in\Gamma_\Phi^t$.

(2) It is easy to check that (a) and (b) are satisfied by Γ_Φ^t. We now prove (c). Let A be a finite subset of Γ generating $\mathbb{Z}^{d+1}$ as a group. By hypothesis, there exists a $\varepsilon>0$ and a $\gamma=(m,\beta)\in\Gamma$ such that $\Phi(m,\beta)\geqslant(t+\varepsilon)m$. It follows that for any $(n,\alpha)\in\Gamma$, we have that

$$\frac{\Phi(km+n,k\beta+\alpha)}{km+n}\geqslant\frac{\Phi(n,\alpha)+k\Phi(m,\beta)}{km+n}\geqslant\frac{\Phi(n,\alpha)+km(t+\varepsilon)}{km+n}\geqslant t$$

for large enough k. There therefore exists a $k_0\geqslant 1$ such that $k\gamma+\xi\in\Gamma_\Phi^t$ for any $\xi\in A$ and $k\geqslant k_0$, so Γ_Φ^t generates $\mathbb{Z}^{d+1}$ as a group.

□

Remark 4.2. The superadditivity of Φ implies that

$$\Delta\left(\Gamma_\Phi^{\varepsilon t_1+(1-\varepsilon)t_2}\right)\supset\varepsilon\Delta(\Gamma_\Phi^{t_1})+(1-\varepsilon)\Delta\left(\Gamma_\Phi^{t_2}\right).$$

By the Brunn–Minkowski theorem, the function $t\mapsto\operatorname{vol}(\Delta(\Gamma_\Phi^t))^{1/d}$ is concave on $]-\infty,\theta[$, where

$$\theta=\lim_{n\to+\infty}\sup_{\alpha\in\Gamma_n}\frac{1}{n}\Phi(n,\alpha),$$

so it is continuous on this interval. Moreover, as the set (dense in $\Delta(\Gamma)$)

$$\{\alpha/n\ :\ (n,\alpha)\in\Gamma,\ n\geqslant 1\}$$

is contained in $\bigcup_{t\in\mathbb{R}}\Delta(\Gamma_\Phi^t)$, we get that

$$\operatorname{vol}(\Delta(\Gamma))=\lim_{t\to-\infty}\operatorname{vol}\left(\Delta(\Gamma_\Phi^t)\right).$$

The following result is a limit theorem for superadditive functions defined on Γ. It is a natural generalisation of [7, Theorem 1.11]

Theorem 4.3. *Let $\Phi : \Gamma \to \mathbb{R}$ be a superadditive function such that*

$$\theta := \lim_{n\to+\infty} \sup_{\alpha\in\Gamma_n} \frac{1}{n}\Phi(n,\alpha) < +\infty. \tag{29}$$

For any integer $n \geqslant 1$, we consider $Z_n = \Phi(n,.)$ as a uniformly distributed random variable on Γ_n. The sequence of random variables $\left(Z_n/n\right)_{n\geqslant1}$ then converges in law[8] to a limit random variable Z whose law is given by

$$\mathbb{P}(Z \geqslant t) = \frac{\mathrm{vol}(\Delta(\Gamma^t_\Phi))}{\mathrm{vol}(\Delta(\Gamma))}, \quad t \neq \theta.$$

Proof. By Remark 4.2 the function F defined on $t \in \mathbb{R}\backslash\{\theta\}$ by $F(t) := \mathrm{vol}(\Delta(\Gamma^t_\Phi))/\mathrm{vol}(\Delta(\Gamma))$ is decreasing and continuous and $\lim\limits_{t\to-\infty} F(t) = 1$. Moreover, condition (29) implies that $F(t) = 0$ for large enough positive t and it follows that if we extend the domain of definition of F to $\mathbb{R}$ by taking $F(\theta)$ to be the limit of $F(t)$ as t tends to θ from the left we get a (left continuous) probability function on $\mathbb{R}$. For any integer $n \geqslant 1$ and any real number t we have that

$$\mathbb{P}(Z_n \geqslant t) = \frac{\#\Gamma^t_{\Phi,n}}{\#\Gamma_n},$$

where $\Gamma^t_{\Phi,n}$ is the set of all $\alpha \in \mathbb{N}^d$ such that $(n,\alpha) \in \Gamma^t_\Phi$. By the previous lemma we have that

$$\lim_{n\to+\infty} \mathbb{P}(Z_n \geqslant t) = F(t) \tag{30}$$

for any $t < \theta$. Moreover, if $t > \theta$ then $\Gamma^t_{\Phi,n}$ is empty for any $n \geqslant 1$ and $\Delta(\Gamma^t_\Phi)$ is also empty, so equation (30) also holds for $t > \theta$. Finally, if the function F is continuous at θ then since both $t \mapsto \mathbb{P}(Z_n \geqslant t)$ and F are decreasing we also have that $\lim\limits_{n\to+\infty} \mathbb{P}(Z_n \geqslant \theta) = F(\theta)$. The result follows. □

Remark 4.4. The limit law in the above theorem can also be characterised as the pushforward of Lesbesgue measure on $\Delta(\Gamma)$ by a function determined by Φ. Let $G_\Phi : \Delta(\Gamma) \to \mathbb{R}\cup\{-\infty\}$ be the map sending x to $\sup\{t \in \mathbb{R} : x \in \Delta(\Gamma^t_\Phi)\}$. This is a real concave function on[9] $\Delta(\Gamma)^\circ$. The function G_Φ is therefore continuous on $\Delta(\Gamma)^\circ$. By definition, the limit law is equal to the pushforward of normalised Lesbesgue measure on $\Delta(\Gamma)$ by G_Φ. In particular, if h is a continuous bounded function then we have that

$$\lim_{n\to+\infty} \frac{h(\Phi(n,\alpha)/n)}{\#\Gamma_n} = \frac{1}{\mathrm{vol}(\Delta(\Gamma))} \int_{\Delta(\Gamma)^\circ} G_\Phi(x)\,\mathrm{vol}(\mathrm{d}x) \tag{31}$$

---

[8] We say that a sequence of random variables $(Z_n)_{n\geqslant1}$ converges in law to a random variable Z if the law of Z_n converges weakly to that of Z, i.e., for any continuous bounded function h on $\mathbb{R}$ we have that $\lim\limits_{n\to+\infty} \mathbb{E}[h(Z_n)] = \mathbb{E}[Z]$, or equivalently, the probability function of Z_n converges to that of Z at any point $x \in \mathbb{R}$ such that $\mathbb{P}(Z = x) = 0$.

[9] The set $\bigcup_{t\in\mathbb{R}} \Delta(\Gamma^t_\Phi)$ is convex and its volume is equal to $\mathrm{vol}(\Delta(\Gamma))$ so it contains $\Delta(\Gamma)^\circ$.

This enables us to realise the random variable Z as the function G_Φ defined on the convex body $\Delta(\Gamma)$ equipped with normalised Lesbesgue measure.

In the rest of this section we apply these results to the situation described in Sect. 4.2. We consider an integral projective scheme X of dimension $d \geqslant 1$ defined on a field k and an invertible $\mathcal{O}_X$-module L. We also choose a regular rational point (it is assumed that such a point exists) $x \in X(k)$, a local system of parameters $(z_1, \ldots, z_d)$ and a monomial order on $\mathbb{N}^d$. Let $V_\bullet$ be a graded linear system on L whose Okounkov semi-group $\Gamma(V_\bullet)$ satisfies[10] conditions (a)-(c) of section 4.3. For any $n \in \mathbb{N}$ let V_n be equipped with two norms φ_n and ψ_n which are assumed to be ultrametric for non-archimedean k.

Theorem 4.5. *Assume the norms φ_n and ψ_n satisfy the following conditions:*

(1) the system of norms $(\varphi_n, \psi_n)_{n\in\mathbb{N}}$ is submultiplicative (i.e. satisfies (27));
(2) we have that $d(\varphi_n, \psi_n) = O(n)$ as $n \to +\infty$;
(3) there is a constant $C > 0$ such that[11] $\inf_{\alpha\in\Gamma(V_n)} \ln \|s_{(n,\alpha)}\|_{\widehat{\varphi}_n} \geqslant -Cn$ for any $n \in \mathbb{N}$, $n \geqslant 1$.

Then the sequence $(\frac{1}{n}\widehat{\mu}(V_n, \varphi_n, \psi_n))_{n\geqslant 1}$ converges in $\mathbb{R}$.

Proof. We introduce auxiliary monomial norms η_n as in Sect. 4.2. Let $\Phi : \Gamma(V_\bullet) \to \mathbb{R}$ be the function that sends $(n, \alpha) \in \Gamma(V_\bullet)$ to $\widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \eta_n)$. This function is superadditive and condition (3) implies that

$$\lim_{n\to+\infty} \sup_{\alpha\in V_n} \frac{1}{n}\Phi(n, \alpha) < +\infty.$$

Let $Z_{\Phi,n} = \Phi(n, .)$ be a uniformly distributed random variable on $\Gamma(V_n)$. By Theorem 4.3 the sequence of random variables $(Z_{\Phi,n}/n)_{n\geqslant 1}$ converges in law to a random variable Z_Φ defined on $\Delta(\Gamma(V_\bullet))$ (as in Remark 4.4). Similarly, conditions (2) and (3) prove that (3) also holds for the norms $\widehat{\psi}_n$. Denote by $\Psi : \Gamma(V_\bullet) \to \mathbb{R}$ the function sending $(n, \alpha) \in \Gamma(V_\bullet)$ to $\widehat{\deg}(\mathrm{gr}^\alpha(V_n), \psi_n, \eta_n)$ and by $Z_{\Psi,n} = \Psi(n, .)$ the random variable on $\Gamma(V_n)$ such that $n \in \mathbb{N}$, $n \geqslant 1$. The sequence of random variables $(Z_{\Psi,n}/n)_{n\geqslant 1}$ then converges in law to a random variable Z_Ψ defined on $\Delta(\Gamma(V_\bullet))$. Moreover, (2) implies that the function $|Z_\Phi - Z_\Psi|$ is bounded on $\Delta(\Gamma(V_\bullet))^\circ$.

By Eq. (28) and the equality

$$\widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \psi_n) = \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \varphi_n, \eta_n) - \widehat{\deg}(\mathrm{gr}^\alpha(V_n), \psi_n, \eta_n),$$

---

[10] Note that these three conditions are automatically satified whenever $V_\bullet$ contains an ample divisor, ie. $V_n \neq \{0\}$ for large enough n and there is an integer $p \geqslant 1$, an ample $\mathcal{O}_X$-module A and a non-zero section s of $pL - A$, such that

$$\mathrm{Im}\big(H^0(X, nA) \xrightarrow{\cdot s^n} H^0(X, npL)\big) \subset V_{np}$$

for any $n \in \mathbb{N}$, $n \geqslant 1$. We refer the reader to [16, lemma 2.12] for a proof.

[11] See Sect. 4.2 for notation.

it will be enough to prove that the sequence $(\mathbb{E}[Z_{\Phi,n}/n]-\mathbb{E}[Z_{\Psi,n}/n])_{n\geqslant 1}$ converges in $\mathbb{R}$. Condition (2) of the theorem implies that the functions $\frac{1}{n}|Z_{\Phi,n}-Z_{\Psi,n}|$ $(n\in\mathbb{N})$ are uniformly bounded. Let $A>0$ be a constant such that

$$\forall\, n\geqslant 1,\quad |Z_{\Phi,n}-Z_{\Psi,n}|\leqslant An.$$

As $(Z_{\Phi,n}/n)_{n\geqslant 1}$ and $(Z_{\Psi,n}/n)_{n\geqslant 1}$ converge in law to Z_Φ and Z_Ψ respectively, for any $\varepsilon>0$ there is a $T_0>0$ and a $n_0\in\mathbb{N}$ such that

$$\forall\, T\geqslant T_0,\ \forall\, n\geqslant n_0,\quad \mathbb{P}(Z_{\Phi,n}\leqslant -nT)<\varepsilon \text{ et } \mathbb{P}(Z_{\Psi,n}\leqslant -nT)<\varepsilon.$$

It follows that

$$\begin{aligned}&\big|\mathbb{E}[Z_{\Phi,n}/n]-\mathbb{E}[Z_{\Psi,n}/n]-\mathbb{E}[\max(Z_{\Phi,n}/n,-T)]+\mathbb{E}[\max(Z_{\Psi,n}/n,-T)]\big|\\&\quad\leqslant 2\varepsilon\mathbb{E}[|Z_{\Phi,n}/n-Z_{\Psi,n}/n|]\leqslant 2\varepsilon A\end{aligned}\tag{32}$$

whenever $T\geqslant T_0$ and $n\geqslant n_0$. Moreover, as the random variables $Z_{\Phi,n}/n$ and $Z_{\Psi,n}/n$ are uniformly bounded above and the sequences $(Z_{\Phi,n}/n)_{n\geqslant 1}$ and $(Z_{\Psi,n}/n)_{n\geqslant 1}$ converge in law it follows that

$$\begin{aligned}&\lim_{n\to+\infty}\mathbb{E}[\max(Z_{\Phi,n}/n,-T)]-\mathbb{E}[\max(Z_{\Psi,n}/n,-T)]\\&\quad=\mathbb{E}[\max(Z_\Phi,-T)-\max(Z_\Psi,-T)].\end{aligned}$$

Moreover, as the function $|Z_\Phi-Z_\Psi|$ is bounded, the dominated convergence theorem implies that

$$\lim_{T\to+\infty}\mathbb{E}[\max(Z_\Phi,-T)-\max(Z_\Psi,-T)]=\mathbb{E}[Z_\Phi-Z_\Psi].$$

Equation (32) then implies that

$$\limsup_{n\to+\infty}\big|\mathbb{E}[Z_{\Phi,n}/n]-\mathbb{E}[Z_{\Psi,n}/n]-\mathbb{E}[Z_\Phi-Z_\Psi]\big|\leqslant 2\varepsilon A.$$

As ε is arbitrary, we get that

$$\lim_{n\to+\infty}\frac{1}{n}\widehat{\mu}(V_n,\varphi_n,\psi_n)=\mathbb{E}[Z_\Phi-Z_\Psi].$$

□

Condition (3) in the above theorem holds whenever φ_n comes from a continuous metric on the invertible $\mathcal{O}_X$-module L. This can be proved by considering a monomial order $\leqslant$ on $\mathbb{N}^d$ such that[12] $\alpha_1+\cdots+\alpha_d<\beta_1+\cdots+\beta_d$ implies $(\alpha_1,\ldots,\alpha_d)<(\beta_1,\ldots,\beta_d)$. Let X^{an} be the analytic space associated to the k-scheme X (in the Berkovich sense [2] if k is non-archimedean) and let L^{an} be the pull-back of L to X^{an}. Let $\mathcal{C}^0_{X^{\mathrm{an}}}$ be the sheaf of continuous real functions on X^{an}. A *continuous metric* on L is a morphism of set sheaves, $\|.\|$, from $L^{\mathrm{an}}\otimes\mathcal{C}^0_{X^{\mathrm{an}}}$ to $\mathcal{C}^0_{X^{\mathrm{an}}}$ which in every point $x\in X^{\mathrm{an}}$ induces a norm $\|.\|(x)$ on the fibre $L^{\mathrm{an}}(x)$.

[12] When $\alpha_1+\cdots+\alpha_d=\beta_1+\cdots+\beta_d$ we may use the lexicographic order, for example.

Given a continuous metric φ on X we can equip $H^0(X, L)$ with the supremum norm $\|.\|_{\varphi,\sup}$ such that

$$\forall\, s \in H^0(X, L), \quad \|s\|_{\varphi,\sup} := \sup_{x\in X^{\mathrm{an}}} \|s\|_\varphi(x).$$

For any integer $n \in \mathbb{N}$ the metric φ induces by passage to the tensor product a continuous metric $\varphi^{\otimes n}$ on nL. Let φ_n be the supremum norm on $H^0(X, nL)$ induced by $\varphi^{\otimes n}$ (or its restriction to V_n by abuse of language): the system of norms $(\varphi_n)_{n\geqslant 0}$ then satisfies condition (3) of Theorem 4.5. This follows from Schwarz's (complex or non-archimedean) Lemma (cf. [9, pp. 205–206]). This gives us the following corollary.

Corollary 4.6. *Let X be a projective integral scheme defined over a field k and let L be an invertible $\mathcal{O}_X$-module equipped with two continuous metrics φ and ψ. Let $V_\bullet$ be a graded linear system of L such that $\Gamma(V_\bullet)$ satisfies conditions (a)–(c) above. For any integer $n \in \mathbb{N}$ let φ_n and ψ_n be the supremum norms on V_n associated to the metrics $\varphi^{\otimes n}$ and $\psi^{\otimes n}$ respectively. The sequence $(\widehat{\mu}(V_n, \varphi_n, \psi_n)/n)_{n\geqslant 1}$ then converges in $\mathbb{R}$.*

Proof. The system of norms $(\varphi_n)_{n\geqslant 0}$ is submultiplicative. If s and s' are elements of V_n and V_m respectively we have that

$$\begin{aligned}\|s\otimes s'\|_{\varphi_{n+m}} &= \sup_{x\in X^{\mathrm{an}}} \|s\otimes s'\|_{\varphi^{\otimes(n+m)}}(x)\\ &\leqslant \Big(\sup_{x\in X^{\mathrm{an}}} \|s\|_{\varphi^{\otimes n}}(x)\Big)\cdot\Big(\sup_{x\in X^{\mathrm{an}}} \|s'\|_{\varphi^{\otimes m}}(x)\Big) = \|s\|_{\varphi_n}\cdot\|s'\|_{\varphi_m}.\end{aligned}$$

Similarly, the system of norms $(\psi_n)_{n\geqslant 0}$ is also submultiplicative. Moreover, as the topological space X^{an} is compact, we have that

$$\sup_{x\in X^{\mathrm{an}}} d(\|.\|_\varphi(x), \|.\|_\psi(x)) < +\infty.$$

and it follows that $d(\varphi_n, \psi_n) = O(n)$ as $n \to +\infty$. Finally, as the norms φ_n satisfy condition (3) of Theorem 4.5, the convergence of $(\widehat{\mu}(V_n, \varphi_n, \psi_n)/n)_{n\geqslant 1}$ as a consequence of this theorem. □

Remark 4.7. This result invites comparison with a result of Witt Nyström's [20, Theorem 1.4]. Both methods use the monomial basis to construct super or subadditive functions on the Okounkov semi-group. However, the method in [20] is based on a comparison between the L^2 metric and the L^∞ metric, whereas we use the Harder–Narasimhan formalism. This new approach is highly flexible and enables us to prove our result in the very general setting of a submultiplicatively normed linear system satisfying moderate conditions, in both the complex and non-archimedean cases.

5. Asymptotic distributions of logarithmic sections

In this section we prove our main theorem.

5.1. A convergence criterium

In this section we prove a convergence criterium. For any real number x the expression x_+ denotes $\max(x, 0)$.

Proposition 5.1. *Let $(Z_n)_{n\geqslant 1}$ be a sequence of uniformly bounded random variables. Assume that for any $t \in \mathbb{R}$ the sequence $(\mathbb{E}[\max(Z_n, t)])_{n\geqslant 1}$ converges in $\mathbb{R}$. The sequence of random variables $(Z_n)_{n\geqslant 1}$ then converges in law.*

Proof. Note that the condition of the proposition implies that, for any $t \in \mathbb{R}$, the sequence $(\mathbb{E}[(Z_n - t)_+])_{n\geqslant 1}$ converges in $\mathbb{R}$. In fact, one has

$$\max(Z_n, t) = (Z_n - t)_+ + t.$$

Let h be a compactly supported smooth function. We have that

$$\mathbb{E}[h(Z_n)] = -\int_{\mathbb{R}} h(t)\,\mathrm{d}\mathbb{P}(Z_n \geqslant t) = \int_{\mathbb{R}} \mathbb{P}(Z_n \geqslant t)h'(t)\,\mathrm{d}t,$$

where the second equality comes from integration by parts. For any $a \in \mathbb{R}$ we have that

$$\int_a^{+\infty} \mathbb{P}(Z_n \geqslant t)\,\mathrm{d}t = \mathbb{E}[(Z_n - a)_+]$$

and it follows that

$$\mathbb{E}[h(Z_n)] = -\int_{\mathbb{R}} h'(t)\,\mathrm{d}\mathbb{E}[(Z_n - t)_+] = \int_{\mathbb{R}} h''(t)\mathbb{E}[(Z_n - t)_+]\,\mathrm{d}t.$$

variables Z_n are uniformly bounded the dominated convergence theorem implies that the sequence $(\mathbb{E}[h(Z_n)])_{n\geqslant 1}$ converges in $\mathbb{R}$. Let $I(h)$ be its limit. The functional $I(.)$ is continuous with respect to the supremum norm on the space of compactly supported smooth functions, so it can be extended by continuity to a positive linear form on the space of continuous compactly supported functions, and hence defines a Radon measure on $\mathbb{R}$. As the random variable Z_n are uniformly bounded it follows that $I(.)$ is a probability measure on $\mathbb{R}$. The result follows. □

5.2. Asymptotic distribution of eigenvalues

In what follows we fix a valued field k which is either $\mathbb{C}$ with the usual absolute value or a complete non-archimedean field. Let X be an integral projective scheme of dimension $d \geqslant 1$ defined over k with a regular rational point. In this subsection we prove the following theorem (cf. Sects. 2.3 and 3.2 for notations).

Theorem 5.2. *Let L be an invertible $\mathcal{O}_X$-module and $V_\bullet$ a graded linear subsystem of L whose Okounkov semi-group satisfies conditions (a)–(c) of Sect. 4.3. Let φ and ψ be two continuous metrics on L, and for any integer $n \geqslant 0$ let φ_n and ψ_n be the supremum norms on V_n induced by the tensor product metrics $\varphi^{\otimes n}$ and $\psi^{\otimes n}$ respectively. Then we have that*

(1) *The sequence of random variables* $(\frac{1}{n}Z_{(V_n,\varphi_n,\psi_n)})_{n\geqslant 1}$ *converges in law to a probability measure on* $\mathbb{R}$.
(2) *the sequence of polygons* $(\frac{1}{n}P_{(V_n,\varphi_n,\psi_n)})_{n\geqslant 1}$ *converges uniformly to a concave function on* $[0,1]$;

Proof. For any $n\in\mathbb{N}$, $n\geqslant 1$ we let Z_n denote the random variable $\frac{1}{n}Z_{(V_n,\varphi_n,\psi_n)}$. We have that

$$|Z_n|\leqslant \sup_{x\in X^{\mathrm{an}}} d(\|.\|_\varphi(x),\|.\|_\psi(x))$$

for any $n\geqslant 1$. The sequence of random variables $(Z_n)_{n\geqslant 1}$ is therefore uniformly bounded. By [10, proposition 1.2.9], the second statement follows from the first.

We now prove the first statement by using the convergence criterion given in 5.1 and the limit result proved in 4.6. For any real parameter a let $\varphi(a)$ be the continuous metric on L such that

$$\forall\, x\in X^{\mathrm{an}},\quad \|.\|_{\varphi(a)}(x)=\mathrm{e}^a\|.\|_\varphi(x).$$

Let $\psi\vee\varphi(a)$ be the metric on L such that

$$\forall\, x\in X^{\mathrm{an}},\quad \|.\|_{\psi\vee\varphi(a)}(x)=\max\big(\|.\|_\psi(x),\|.\|_{\varphi(a)}(x)\big).$$

The supremum norm on V_n associated to the metric $(\psi\vee\varphi(a))^{\otimes n}$ is $\psi_n\vee\varphi_n(an)$ (with the notations as in Sects. 2.5 or 3.3). Corollary 4.6 applied to φ and $\psi\vee\varphi(a)$ proves that the sequence $(\widehat{\mu}(V_n,\varphi_n,\psi_n\vee\varphi_n(na))/n)_{n\geqslant 1}$ converges in $\mathbb{R}$. Moreover, by Propositions 2.8 and 3.7 we have that

$$\left|\frac{1}{n}\widehat{\mu}(V_n,\varphi_n,\psi_n\vee\varphi_n(na))-\mathbb{E}[\max(Z_n,a)]\right|\leqslant\frac{1}{n}A(r_n),$$

where $r_n=\mathrm{rk}(V_n)$ and

$$\forall\, r\in\mathbb{N},\ r\geqslant 1,\quad A(r):=2\ln(r)+\frac{1}{2}\ln(2).$$

As $r_n=O(n^d)$ when $n\to+\infty$, we have that $\lim_{n\to+\infty}A(r_n)/n=0$. It follows that the sequence $(\mathbb{E}[\max(Z_n,a)])_{n\geqslant 1}$ converges. By Proposition 5.1, the result follows. □

Remark 5.3. The above result still holds whenever we replace φ_n and ψ_n by norms φ_n' and ψ_n' such that

$$\max(d(\varphi_n,\varphi_n'),d(\psi_n,\psi_n'))=o(n),\quad n\to+\infty,$$

and the limit laws are the same. If we let Z_n' be the random variable $\frac{1}{n}Z_{(V_n,\varphi_n',\psi_n')}$ then for any $a\in\mathbb{R}$ we have that

$$\begin{aligned}&\big|\mathbb{E}[\max(Z_n,a)]-\mathbb{E}[\max(Z_n',a)]\big|\\ &\quad\leqslant\frac{1}{n}\big|\widehat{\mu}(V_n,\varphi_n,\psi_n\vee\varphi_n(an))-\widehat{\mu}(V_n,\varphi_n',\psi_n'\vee\varphi_n(an))\big|+\frac{1}{n}A(r_n)\\ &\quad\leqslant\frac{1}{n}(d(\varphi_n,\varphi_n')+d(\psi_n,\psi_n')+A(r_n)).\end{aligned}$$

Letting $n \to +\infty$ we get that

$$\lim_{n\to+\infty} \mathbb{E}[\max(Z'_n, a)] = \lim_{n\to+\infty} \mathbb{E}[\max(Z_n, a)].$$

In particular, this enables us to apply the theorem to L^2 norms when k is the field of complex numbers—see for example Lemma 3.2 of [3].

The convergence of the sequence of polyones in the above theorem implies, in considering the convergence at the point 1, the following local analogue of the arithmetic Hilbert–Samuel theorem.

Corollary 5.4. *With the notation and the hypothese of the above theorem, the limite*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(V_n, \varphi_n, \psi_n)}{n^{d+1}}$$

exists in $\mathbb{R}$.

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References

[1] Abbes, A., Bouche, T.: Théoréme de Hilbert–Samuel arithmétique. Université de Grenoble. Ann. de l'Inst. Fourier **45**(2), 375–401 (1995)

[2] Berkovich, V.G.: Spectral theory and analytic geometry over non-Archimedean fields, mathematical surveys and monographs, vol. 33. American Mathematical Society, Providence (1990)

[3] Berman, R., Boucksom, S.: Growth of balls of holomorphic sections and energy at equilibrium. Invent. Math. **181**(2), 337–394 (2010)

[4] Berndtsson, B.: Probability measures associated to geodesic in the space of Kähler metrics, preprint (2009)

[5] Bost, J.-B.: Périodes et isogénies des variétés abéliennes sur les corps de nombres (d'après D. Masser et G. Wüstholz). Astérisque no. 237, p. Exp. No. 795, 4, pp. 115–161, Séminaire Bourbaki, Vol. 1994/1995 (1996)

[6] Boucksom, S.: Corps d'Okounkov (d'après Andri Okounkov, Robert Lazarsfeld et Mircea Mustaţă, Kiumars Kavech et Askold Khovanskii). Séminaire Bourbaki, vol. 2011/2012 (to appear)

[7] Boucksom, S., Chen, H.: Okounkov bodies of filtered linear series. Compos. Math. **147**(4), 1205–1229 (2011)

[8] Bourbaki, N.: Espaces vectoriels topologiques. Chapitres 1 à 5, new, Masson, Paris, Éléments de mathématique (1981)

[9] Chambert-Loir, A.: Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky). Astérisque no. 282, p. Exp. No. 886, viii, pp. 175–209, Séminaire Bourbaki, vol. 2000/2001 (2002)

[10] Chen, H.: Convergence des polygones de Harder–Narasimhan. Mém. Soc. Math. France **120**, 1–120 (2010)
[11] Chen, H.: Harder–Narasimhan categories. J. Pure Appl. Algebra **214**(2), 187–200 (2010)
[12] Eisenbud, D.: Commutative algebra. Graduate texts in mathematics, vol. 150. Springer, New York, With a view toward algebraic geometry (1995)
[13] Gaudron, É.: Pentes de fibrés vectoriels adéliques sur un corps globale. Rendiconti del Seminario Matematico Della Università di Padova **119**, 21–95 (2008)
[14] Gaudron, É.: Minorations simultanées de formes linéaires de logarithmes de nombres algébriques. Bull. Soc. Math. France **142**(1), 1–62 (2014)
[15] Kaveh, K., Khovanskii, A.: Algebraic equations and convex bodies. In: Perspectives in Analysis, Geometry, and Topology, Progr. Math., vol. 296, pp. 263–282. Birkhäuser/Springer, New York (2012)
[16] Lazarsfeld, R., Mustaţă, M.: Convex bodies associated to linear series. Ann. Sci. l'École Normale Supér. Quatr. Série **42**(5), 783–835 (2009)
[17] Okounkov, A.: Brunn–Minkowski inequality for multiplicities. Invent. Math. **125**(3), 405–411 (1996)
[18] Perez-Garcia, C., Schikhof, W.H.: Locally convex spaces over non-Archimedean valued fields, Cambridge Studies in Advanced Mathematics, vol. 119. Cambridge University Press, Cambridge (2010)
[19] Thompson, A.C.: Minkowski geometry, Encyclopedia of Mathematics and its Applications, vol. 63. Cambridge University Press, Cambridge (1996)
[20] Witt Nyström, D.: Transforming metrics on a line bundle to the Okounkov body. Ann. Sci. l'École Normale Supér. Quatr Série **47**(6) (2014)